TECHNOLOGY DEPT.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 63 NUMBER 6

CONTENTS

The Basic Concepts of Algebraic Logic . . . PAUL R. HALMOS 363
Zero Integral Homogeneous Symmetric Functions in Certain Rings . . . . . MILO WEAVER 387
Stacking Colored Cubes . . . . . PAUL B. JOHNSON 392
A National Weakness . . . . . JOHN W. GARDNER 396
Mathematical Notes . . . . . JOHN CHRISTOPHER, E. P. MILES, JR. AND ERNEST WILLIAMS, H. S. THURSTON, S. M. SHAH, A. I. MARTIN, H. K. FARAHAT AND L. MIRSKY 399
Classroom Notes . . . J. G. CAMPBELL, R. C. BUCK, O. J. KARST DIRAN SARAFYAN, W. R. UTZ 413
Elementary Problems and Solutions 420
Advanced Problems and Solutions 426
Recent Publications 433
News and Notices 442
The Mathematical Association of America 446
The New Editor-in-Chief 446
New Members 447
The January Meeting of the Northern California Section . . . 449
The February Meeting of the Louisiana-Mississippi Section . . 452

JUNE-JULY 1956

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

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THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.
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Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year. To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash., during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

THE BASIC CONCEPTS OF ALGEBRAIC LOGIC

PAUL R. HALMOS, University of Chicago

1. Introduction. It has often happened that a theory designed originally as a tool for the study of a physical problem came subsequently to have purely mathematical interest. When that happens, the theory is usually generalized way beyond the point needed for applications, the generalizations make contact with other theories (frequently in completely unexpected directions), and the subject becomes established as a new part of pure mathematics. The part of pure mathematics so created does not (and need not) pretend to solve the physical problem from which it arises; it must stand or fall on its own merits.

Physics is not the only external source of mathematical theories; other disciplines (such as economics and biology) can play a similar role. A recent (and possibly somewhat surprising) addition to the collection of mathematical catalysts is formal logic; the branch of pure mathematics that it has precipitated will here be called *algebraic logic*.

Algebraic logic starts from certain special logical considerations, abstracts from them, places them into a general algebraic context, and, *via* the generalization, makes contact with other branches of mathematics (such as topology and functional analysis). It cannot be overemphasized that algebraic logic is more algebra than logic. Algebraic logic does not claim to solve any of the vexing foundation problems that sometimes occupy logicians. All that is claimed for it is that it is a part of pure mathematics in which the concepts that constitute the skeleton of modern symbolic logic can be discussed in algebraic language. The discussion serves to illuminate and clarify those concepts and to indicate their connection with ordinary mathematics. Whether the subject as a whole will come to be considered sufficiently interesting and sufficiently deep to occupy a place among pure mathematical theories remains to be seen.

The literature of algebraic logic is not yet very extensive, and the few items that are available are highly technical. It is for that reason that this expository paper was written; its main purpose is to kindle interest in a young but promising subject. In such a context it does not seem to be appropriate to burden the reader with the usual scholarly references and assignments of credit. At the very least, however, the names of the principal contributors should be mentioned. Here they are: Curry, Henkin, Rasiowa, Sikorski, and Tarski. Many of the ideas of algebraic logic have been in the air for several years and were known, at least subconsciously, by most logicians. The greatest contributions are those of Tarski; especially relevant is his work (with Jónsson) on Boolean algebras with operators and his theory of cylindric algebras. (Most of the latter material is unfortunately unpublished.) The reader who wishes to study the details will find exact references in two papers on algebraic logic by the present author; the first is in *Compositio Mathematica* (1955), and the second is to appear in *Fundamenta Mathematicae* (1957).

2. Boolean algebras. The father of algebraic logic is George Boole; it is appropriate that a discussion of the subject begin with a quick review of the algebras that bear his name. The shortest definition of Boolean algebras is in terms of the theory of rings: a *Boolean algebra* is a ring with unit in which every element is idempotent (*i.e.*, if p is in the ring, then $p^2=p$). The simplest example, and one that plays a vitally important role throughout the theory, is the field of integers modulo 2; this Boolean algebra will be denoted by **O**.

Boolean algebras have an almost embarrassingly rich structure. It is, for instance, an easy consequence of the definition that a Boolean algebra always has characteristic 2 (*i.e.*, $p+p=0$) and that as a ring it is always commutative (*i.e.*, $pq=qp$). In every Boolean algebra there is, moreover, a natural order relation; it is defined by writing $p\leqq q$ if and only if $pq=p$. The algebraic structure and the order structure are as compatible as they can be. The algebraic zero is also the order zero (*i.e.*, the least element of the algebra), and the algebraic unit is also the order unit (*i.e.*, the greatest element of the algebra); in other words, $0\leqq p\leqq 1$ for every p. With respect to the order, the algebra turns out to be a complemented lattice. The lattice operations can be expressed in terms of the given algebraic operations, as follows: the complement of p, denoted by p', is $1+p$; the infimum of p and q, denoted by $p\wedge q$, is pq; and the supremum of p and q, denoted by $p\vee q$, is $p+q+pq$.

The process of defining many useful operations and relations in a Boolean algebra, in terms of addition and multiplication, is to a large extent reversible. This fact is responsible for the abundance of different axiomatic approaches to the subject. A Boolean algebra can be defined in terms of its partial order, or in terms of complements and suprema, or in terms of complements and infima, and so on and so forth *ad* almost *infinitum*. Thus, for instance, since sums and products can be expressed in terms of complements and infima ($pq=p\wedge q$ and $p+q=(p'\wedge q')'\wedge(p\wedge q)'$), it follows that if a set admits a unary operation and a binary operation satisfying the appropriate conditions, then that set is a Boolean algebra. (It isn't really, but it is close enough to make the distinction pedantic. What should be said is that if addition and multiplication are defined as indicated above, then the set, together with the defined operations, constitutes a Boolean algebra.) The appropriate conditions are simple to describe. They require that the underlying set contain at least two distinct elements, and that

(I 1) $$p\wedge(q\wedge r)=(p\wedge q)\wedge r,$$

(I 2) $$p\wedge q=q\wedge p,$$

(I 3) $$\text{if } p\wedge q'=r\wedge r', \text{ then } p\wedge q=p,$$

(I 4) $$\text{if } p\wedge q=p, \text{ then } p\wedge q'=r\wedge r',$$

for all elements p, q, and r. (Caution: (I 3) means that if $p\wedge q'=r\wedge r'$ for *some* r, then $p\wedge q=p$, and (I 4) means that if $p\wedge q=p$, then $p\wedge q'=r\wedge r'$ for *all* r.)

Since, inside a fixed non-empty set, set-theoretic complementation and the formation of set-theoretic intersections do satisfy these conditions, any class of sets that is closed under these two operations is a Boolean algebra. The class of all subsets of a non-empty set is the most easily described (but far from the only) example of a Boolean algebra obtained in this way.

[Terminological purists sometimes object to the Boolean use of the word "algebra." The objection is not really cogent. In the first place, the theory of Boolean algebras has not yet collided, and it is not likely to collide, with the theory of linear algebras. In the second place, a collision would not be catastrophic; a Boolean algebra is, after all, a linear algebra over the field of integers modulo 2. The last, but not the least, pertinent comment is a pragmatic one. While, to be sure, a shorter and more suggestive term than "Boolean algebra" might be desirable, the nomenclature is so thoroughly established that to change now would do more harm than good.]

It is amusing and instructive to compare the axiom system (I) with the following, somewhat unorthodox, system of axioms for groups. A group may be defined as a non-empty set with a unary operation $(p \to p^-)$ and a binary operation $((p, q) \to p \times q)$ such that

(II 1) $$p \times (q \times r) = (p \times q) \times r,$$

(II 2) $$(p \times q)^- = q^- \times p^-,$$

(II 3) $$\text{if } p \times q = r \times r^-, \text{ then } p = q^-,$$

(II 4) $$\text{if } p = q^-, \text{ then } p \times q = r \times r^-,$$

for all elements p, q, and r. (Caution: (II 3) means that if $p \times q = r \times r^-$ for *some* r, then $p = q^-$, and (II 4) means that if $p = q^-$, then $p \times q = r \times r^-$ for *all* r.) It is clear that if, in a group, p^- is defined to be the inverse of p, and $p \times q$ is defined to be the product of p and q (in that order), then the conditions (II) are satisfied. The fact that, conversely, the conditions (II) are characteristic of inversion and multiplication in groups is an easy exercise in elementary axiomatics.

[One comment should be made on the independence of the axiom set (II). The fact is that the set is *not* independent; the first three axioms are sufficient to characterize groups, and, in particular, they imply the fourth. The reason (II) is offered in the present form is to emphasize its similarity with the axiom set (I) for Boolean algebras. Each of the axioms (II 1), (II 2), and (II 3) is independent of the other three axioms of the set (II).]

3. Propositional calculi. To understand the connection between Boolean algebra and logic, a good way to begin is to examine how sentences are combined by means of sentential connectives. To ensure an unprejudiced approach to the subject, it is desirable to proceed as abstractly as possible. Suppose, therefore, that there is given an arbitrary non-empty set S; intuitively the elements of S are to be thought of as the basic sentences of some theory that is being studied. Suppose, moreover, that A and N are two distinct objects not contained in S; intuitively A and N are to be thought of as the connectives "and" and "not."

(Warning: in other expositions, A is frequently used for the dual connective "or".) Consider finite sequences whose terms are either elements of S or else A or N. (The empty sequence is not included.) If s is a sequence of length n, say, so that $s_0, \cdots, s_{n-1}$ are elements of $S \cup \{A, N\}$, let Ns be the sequence defined by

$$(Ns)_0 = N, \qquad (Ns)_i = s_{i-1} \qquad (i = 1, \cdots, n);$$

if s and t are sequences (of lengths n and m respectively), let Ast be the sequence defined by

$$(Ast)_0 = A, \quad (Ast)_i = s_{i-1} \ (i = 1, \cdots, n), \quad (Ast)_{n+j} = t_{j-1} \ (j = 1, \cdots, m).$$

Let S^* be the smallest set of sequences such that (1) if s is a sequence of length 1 whose unique term is in S, then $s \in S^*$, (2) if $s \in S^*$, then $Ns \in S^*$, and (3) if $s \in S^*$ and $t \in S^*$, then $Ast \in S^*$. In other words, S^* is the set of sequences generated from the one-term sequences of S by means of the operations of prefixing N to a sequence and prefixing A to the concatenation of two sequences. Intuitively S^* is to be thought of as the set of sentences generated from the basic sentences by the two basic connectives. The device of writing Ast instead of $s\ A\ t$ (an ingenious Polish invention) is designed to avoid the multiple layers of parentheses that the more intuitive infix notation necessitates.

The set S^* by itself is not quite a proper object of logical study. The trouble is that if, for instance, s and t are in S^*, then Ast and Ats are distinct elements of S^*, whereas common sense seems to demand that if s and t are sentences, then "s and t" and "t and s" should be sentences that "say the same thing" in some sense. Sentences admit, in other words, a natural equivalence relation; such a relation should therefore be introduced into S^*. A little thought about the intuitive interpretation of the elements of S^* will suggest many conditions that equivalence should satisfy. If the assertion that the elements s and t of S^* are equivalent is denoted by $s \equiv t$, then, for all elements s, t, and u of S^*, it should at the very least be required that

(III 1) $$AsAtu \equiv AAstu,$$

(III 2) $$Ast \equiv Ats,$$

(III 3) $$\text{if } AsNt \equiv AuNu, \text{ then } Ast \equiv s,$$

(III 4) $$\text{if } Ast \equiv s, \text{ then } AsNt \equiv AuNu,$$

(III 5) $$\text{if } s \equiv t, \text{ then } Ns \equiv Nt,$$

(III 6) $$\text{if } s \equiv t, \text{ then } Asu \equiv Atu.$$

In addition, of course, it is necessary that the concept of equivalence used here be an honest equivalence, *i.e.*, that it be reflexive, symmetric, and transitive.

There are likely to be many equivalence relations satisfying the conditions (III); one such is defined by writing $s \equiv t$ for all s and t. In order to avoid this triviality (and for other reasons), it is desirable to consider the smallest possible

equivalence (*i.e.*, the one in which the fewest possible pairs turn out to be equivalent) satisfying these conditions. This makes sense. Indeed, if an equivalence relation is thought of as a certain set of ordered pairs, then the intersection of all the equivalence relations satisfying (III) is an equivalence relation satisfying (III). If, from now on, the symbol $\equiv$ is used to denote this minimal equivalence, then the pair $(S^*, \equiv)$ is one of the simplest non-trivial logical structures; it is usually known as the *propositional* (or *sentential*) *calculus*. There are really many propositional calculi; there is one corresponding to each non-empty set S. It is clear, however, that the only thing that matters (all other differences between the various propositional calculi being essentially notational matters) is the cardinal number of S. It is customary (but not particularly profitable) to assume that S is countably infinite.

4. Axioms and rules. The time has come to make the notation more transparent. While the symbols A and N are technically convenient (no parentheses), it is usually a cumbersome job to decode the sentences involving them and to recapture their intended intuitive content. Accordingly, in what follows, $(s)'$ will be used as an alternative symbol for Ns, and, similarly, $(s) \wedge (t)$ will be used as an alternative symbol for Ast. Thus, for instance, $AsNt$ can be denoted by $(s) \wedge ((t)')$; with the usual mathematical conventions about omitting superfluous parentheses, this becomes $s \wedge t'$. It should now be clear that the conditions (III 1)–(III 4) are only notationally different from (I 1)–(I 4); the conditions (III 5) and (III 6) assert, in customary mathematical language, that $=$ is a congruence relation with respect to the operations of attaching $'$ and infixing $\wedge$.

The equivalence relations that occur in algebra (*e.g.*, the congruence relations in a ring) are usually described by specifying a particular equivalence class (the kernel) and a particular operation (subtraction); it then turns out that two elements are congruent if and only if their difference belongs to the kernel. A similar procedure is available to define the equivalence $\equiv$ described above. In order to motivate the choice of the kernel and the choice of the pertinent subtraction operation, consider first of all an element of S^* that has the form $s \wedge s'$. If the elements of S^* are interpreted as sentences, then there is something obviously undesirable about such an element; in some intuitive sense it is "false." By the same token, the result of attaching $'$ to such an element converts it into something quite laudable; sentences such as that are "true." The kernel that is usually considered is the equivalence class of any particular "true sentence" so obtained. It is pleasant to be able to report that this kernel is independent of the arbitrary choice of its generator. The equivalence class of any element of the form $(s \wedge s')'$ contains all other elements of that form, and, in fact, it consists exactly of all the elements that common sense would declare to be "true." An element of this kernel is called a *tautology* of the propositional calculus. The subtraction operation is the one that associates with two elements s and t of S^* the element $(s \wedge t')' \wedge (s' \wedge t)'$. (In the original notation this reads $ANAsNtNANst$.) The perceptive reader will note that if s and t are interpreted as sentences, then the intuitive interpretation of the proposed "difference" between s and t is the

sentence "s if and only if t." This subtraction does what it was tacitly promised it would do; it is true that a necessary and sufficient condition that s be equivalent to t is that the difference of s and t belong to the kernel. (In customary logical language: a necessary and sufficient condition for $s \equiv t$ is that the biconditional of s and t be a tautology. It is a happy circumstance that biconditioning is a commutative operation; the order of s and t is immaterial.)

In principle it is clear how the procedure used above could be reversed. The tautologies could be defined by making a complete list of them, and equivalence could then be defined in terms of tautologies and the formation of biconditionals. The list of tautologies would be infinite, to be sure, but a clever classifier might nevertheless succeed in describing it in a finite number of words. Something like this is what is usually done. In most text-book presentations of the propositional calculus one finds a small list of special tautologies, together with a few easy instructions for manufacturing others; a general tautology is then defined to be a member of the subset of S^* generated from the special tautologies by repeated applications of the instructions. The given special tautologies (whose particular choice is largely a matter of individual taste) are called *axioms*, and the instructions for manufacturing others are called *rules of inference*. This "axiomatic" procedure (which has become traditional) has no particular advantages (or disadvantages) in comparison with the one followed above; its final result is merely an alternative description of the propositional calculus.

[Here, for the sake of completeness, are the details of one of the popular axiomatic approaches to the propositional calculus. Let $s \vee t$ and $s \rightarrow t$ be abbreviations for $(s' \wedge t')'$ and $(s \wedge t')'$ respectively (or, in the original notation, for $NANsNt$ and $NAsNt$ respectively); the axioms are most conveniently, and most understandably, described by means of these abbreviations. The axioms consist of all those elements of S^* that are of one of the following four forms:

(1) $$(s \vee s) \rightarrow s,$$

(2) $$s \rightarrow (s \vee t),$$

(3) $$(s \vee t) \rightarrow (t \vee s),$$

(4) $$(s \rightarrow t) \rightarrow ((u \vee s) \rightarrow (u \vee t)),$$

where s, t, and u are arbitrary elements of S^*. There is only one rule of inference (called *modus ponens*). According to that rule, if s and t are elements of S^* such that both s and $s \rightarrow t$ are tautologies, then t is a tautology.]

5. Free Boolean algebras. By whatever method the pair $(S^*, \equiv)$ is obtained, once it is at hand it is natural to form the set $\mathbf{A}^*$ of all equivalence classes. In analogy with calling an element of S^* a sentence, an element of $\mathbf{A}^*$ may be called a *proposition*. (The word "proposition" is not always defined this way. The intuitive reasons for using the definition are obvious, but, admittedly, they are not overwhelming.) The set $\mathbf{A}^*$ of propositions possesses, in a natural way, the structure of a Boolean algebra. If $p \in \mathbf{A}^*$, let s be any element of the equivalence class p and write p' for the equivalence class of Ns (*i.e.*, of s'). If both s_1 and s_2

belong to p, *i.e.*, if $s_1 \equiv s_2$, then, by (III 5), $Ns_1 \equiv Ns_2$, so that the definition of p' is unambiguous. If p and q are in $\mathbf{A}^*$, let s and t be corresponding representative elements (*i.e.*, $s \in p$ and $t \in q$) and write $p \wedge q$ for the equivalence class of Ast (*i.e.*, of $s \wedge t$). The necessary unambiguity argument is based on (III 6) this time. The fact that the conditions (I) are satisfied now follows immediately from a comparison of those conditions with the corresponding conditions (III).

The procedure for arriving at $\mathbf{A}^*$ is familiar to anyone who ever heard of the theory of free groups; the Boolean algebra $\mathbf{A}^*$ is, in fact, isomorphic to the free Boolean algebra generated by the originally given set S. In logical studies it is customary to describe the algebraic properties of $\mathbf{A}^*$ by some rather specialized terminology. Thus, for instance, the fact that $\mathbf{A}^*$ satisfies the first necessary condition for being a Boolean algebra (the possession of at least two distinct elements) is usually described by saying that the propositional calculus is consistent.

In view of what has already been said, it is not at all surprising that the construction of $\mathbf{A}^*$ has a very near parallel in group theory. Given an arbitrary non-empty set S, form the set S^* exactly as before and introduce into S^* the smallest equivalence relation $\equiv$ such that

(IV 1) $$AsAtu \equiv AAstu,$$

(IV 2) $$NAst \equiv ANtNs,$$

(IV 3) $$\text{if } Ast \equiv AuNu, \text{ then } s \equiv Nt,$$

(IV 4) $$\text{if } s \equiv Nt, \text{ then } Ast \equiv AuNu,$$

(IV 5) $$\text{if } s \equiv t, \text{ then } Ns \equiv Nt,$$

(IV 6) $$\text{if } s \equiv t, \text{ then } Asu \equiv Atu,$$

for all elements s, t, and u of S^*. The set of all equivalence classes possesses, in a natural way, the structure of a group. The obvious details may safely be omitted; suffice it to say that the proof depends on a comparison of (II) with (IV). The group so obtained is, in fact, isomorphic to the free group generated by the originally given set S.

[The method of sequences-*cum*-equivalence is essentially the only known way of constructing free groups. It is worth noting that for Boolean algebras the following much more elegant method is available. Given an arbitrary (possibly empty) set S, let X be the set of all subsets of S. For each s in S, let $p(s)$ be the set of all those elements of X that contain s. Let $\mathbf{A}^*$ be the Boolean algebra generated by all the sets of the form $p(s)$ with s in S, so that $\mathbf{A}^*$ is a Boolean subalgebra of the Boolean algebra of all subsets of X. Assertion: $\mathbf{A}^*$ is isomorphic to the free Boolean algebra generated by S. The proof of the assertion is an easy exercise in set theory. Unfortunately this method of constructing $\mathbf{A}^*$ is quite provincial; it works like a charm for Boolean algebras, but it does not work for very many algebraic systems, at least not without significant modifications.]

6. Quotients of free algebras. Since the algebraic end-product of the propositional calculus is a free Boolean algebra, that calculus, as it now stands, is too special for most logical purposes. (Freedom for an algebraic system means, roughly speaking, that the only relations among its elements are the trivial ones). The difficulty is visible in the starting point of the construction of the propositional calculus; its source is the fact that the generating set S is a set with no algebraic structure at all. In realistic situations the basic sentences of a theory are likely to be related to each other in various intricate ways, and, in fact, it is usually considered to be one of the chief functions of logic to study such relations and their consequences.

Suppose, for instance, that someone wants to undertake a logical examination of a fragment of the history of English literature. Among the basic sentences of the theory (*i.e.*, among the elements of S) there might occur "Shakespeare wrote Hamlet" and "Bacon wrote Macbeth"; call these sentences s_0 and t_0 respectively. ("Basic sentence" is not the same as "axiom." The sentences s_0 and t_0 need not, at first, be considered either "true" or "false"; they are just considered.) Under these circumstances the investigator would perhaps want to declare the sentence NAs_0t_0 to be an axiom. Since, however, NAs_0t_0 is not a tautology of the propositional calculus, the word "axiom" must be used here in a sense broader than the one discussed above. Such a broadening is perfectly feasible. One way to achieve it is to add another condition to the set (III); the new condition could, for instance, be

(III 7) $$As_0t_0 \equiv As_0Ns_0.$$

The effect of this adjunction is to enlarge the original equivalence relation; the new equivalence relation is not the smallest relation satisfying (III 1)–(III 6), but the smallest relation satisfying (III 1)–(III 7). The same purpose can be accomplished by adjoining to the axioms of the propositional calculus the *extra-logical axiom*

$$NAs_0t_0,$$

and then proceeding exactly as before. The result, with either approach, is that each new equivalence class is the union of several old ones. The tautologies, in particular, all belong to the same (new) equivalence class; an element of this equivalence class is usually called a *provable sentence* or a *theorem* of the modified propositional calculus. If there are at least two (new) equivalence classes (in logical language: if the adjoined axioms are consistent), the set **A** of all such classes possesses in a natural way the structure of a Boolean algebra, which, in general, is not free.

The "free" procedure for constructing non-free Boolean algebras has its group-theoretic parallel. One way to ensure that the generators of a group satisfy certain prescribed relations is to build those relations into the equivalence by means of which the quotient system (S^* modulo $\equiv$) is formed. The disadvantage

of such a procedure is that it is repetitious; it uses the method of an earlier construction instead of its result. It is more usual, and algebraically more satisfactory, to apply the combinatorial method only once; thereafter, all groups defined by generators and relations are constructed by forming a quotient group of one of the free groups already obtained. This works very smoothly; the main reason it works is that every group is a quotient group of a free group.

Since, similarly, every Boolean algebra is a quotient algebra of a free Boolean algebra, the group-theoretic shortcut is available for Boolean algebras too. Just as in the theory of groups, moreover, the core of the idea is applicable to algebras that are not necessarily free. It makes sense (and it is often useful) to force the elements of a Boolean algebra (or group) to satisfy some relations, even if the algebra (or group) is not free to start with. A well-known example is the process of reducing a group by its commutator subgroup and thereby forcing it to become abelian.

7. Filters and ideals. From the point of view of logic, the reduction of Boolean algebras is connected with the theory of provability. It turns out that from the point of view of algebraic logic the most useful approach to that theory is not to ask "What is a proof?" or "How does one prove something?" but to ask about the structure of the set of provable propositions. Suppose, therefore, that **A** is a Boolean algebra, whose elements are to be thought of, intuitively speaking, as the propositions of some theory, and suppose that a non-empty subset **P** of **A** has been singled out somehow; the elements of **P** are to be thought of as the provable propositions of the theory. Common sense suggests that if s and t are provable sentences, then "s and t" is a provable sentence, and if s is a provable sentence, then "s or t" is a provable sentence, no matter what the sentence t may be. In order to meet these demands of common sense, the set **P** cannot be arbitrary; it must be such that if both p and q belong to **P**, then $p \wedge q$ belongs to **P**, and if p belongs to **P**, then $p \vee q$ belongs to **P** for all q in **A**. If a non-empty subset of a Boolean algebra satisfies these conditions, it is called a *filter*. An illuminating comment is this: a necessary and sufficient condition that a subset **P** of a Boolean algebra be a filter is that $1 \in \mathbf{P}$ (*i.e.*, all tautologies are provable), and that if $p \in \mathbf{P}$ and $(p \rightarrow q) \in \mathbf{P}$ (recall that $(p \rightarrow q) = (p' \vee q)$), then $q \in \mathbf{P}$ (*i.e.*, *modus ponens* is a rule of inference).

A filter is not a commonly encountered mathematical object, but one of its first cousins (namely, an ideal) is known to every mathematician. A non-empty subset **M** of a Boolean algebra is an *ideal* (for occasional emphasis, a *Boolean ideal*) if it contains $p \vee q$ whenever it contains both p and q and if it contains $p \wedge q$ whenever it contains p. Although it looks slightly different, this definition is, in fact, equivalent to the usual one; a Boolean algebra is, after all, a ring, and a Boolean ideal in the present sense is the same as an ordinary algebraic ideal.

Each of the two concepts (filter and ideal) is, in a certain sense, the Boolean dual of the other. (Some authors indicate this relation by using the term *dual-ideal* instead of filter.) Specifically, if **P** is a filter in a Boolean algebra **A**, and if

M is the set of all those elements p of **A** for which $p' \in \mathbf{P}$, then **M** is an ideal in **A**; the reverse procedure (making a filter out of an ideal) works similarly. This comment indicates the logical role of the algebraically more common concept; just as filters arise in the theory of provability, ideals arise in the theory of refutability. (A proposition p is called refutable if its negation p' is provable.) Duality is so ubiquitous in Boolean theory that every development of that theory (unless it is exactly twice as long as it should be) must make an arbitrary choice between two possible approaches. Logic is usually studied from the "1" approach, *i.e.*, the emphasis is on truth and provability, and, consequently, on filters. Since the dual "0" approach uses the algebraically more natural concept of ideal, the remainder of this exposition (addressed to mathematicians, rather than to professional logicians) will be couched in terms of the logically less pleasant concepts of falsehood and refutability.

8. Boolean logics. The preceding discussion was intended to motivate the following definition. A *Boolean logic* is a pair (**A**, **M**), where **A** is a Boolean algebra and **M** is a Boolean ideal in **A**. The elements of **A** will be called propositions; the elements of **M** will be called refutable propositions. The group-theoretic analogue of this concept (the structure consisting of a group and a specified normal subgroup) has not received very much attention, but it would strike most algebraists as a perfectly reasonable object of study.

The concept of a Boolean logic (**A**, **M**) has many similarities with the earlier concept of the propositional calculus (S^*, $\equiv$). Both objects consist of (1) a set already endowed with some structure, and (2) a congruence relation in that set. (In Boolean theory, as in the rest of algebra, there is a natural one-to-one correspondence between ideals and congruence relations.) It should not be surprising therefore that the "axiomatic" method is the most common way of converting a Boolean algebra into a Boolean logic. In algebraic terms the axiomatic method amounts simply to this: given **A**, select an arbitrary subset $\mathbf{M}_0$ of **A**, and let **M** be the ideal generated by $\mathbf{M}_0$. (Because **M** consists of the refutable propositions, not the provable ones, the elements of $\mathbf{M}_0$ are "anti-axioms"; their negations are axioms in the usual sense.) Even this procedure has its group-theoretic analogue; for an example, recall the usual definition of the commutator subgroup of a group.

Most logical concepts have an algebraic alter-ego definable within the theory of Boolean logics. Two concepts of special importance (consistency and completeness) will be used to illustrate this point.

A Boolean logic (**A**, **M**) is called *consistent* if for no proposition p in **A** are both p and p' provable, or, equivalently, if for no p in **A** do both p and p' belong to **M**. Since **M** is an ideal, it follows that (**A**, **M**) is consistent if and only if the ideal **M** is proper. To say that a Boolean logic is consistent is to say, roughly speaking, that the set of propositions that are refutable in it is not too large.

On intuitive (pragmatic) grounds it is desirable that the set of refutable propositions be not too small. (Recall that there is a natural one-to-one correspondence between refutable propositions and provable propositions.) The

simplest way to ensure that the set of refutable propositions is large enough is to insist that, for every proposition p, either p or p' be refutable. A Boolean logic satisfying this condition is called *complete*. In other words, $(\mathbf{A}, \mathbf{M})$ is complete if and only if, for every p in $\mathbf{A}$, either $p \in \mathbf{M}$ or $p' \in \mathbf{M}$.

Inconsistent Boolean logics, *i.e.*, logics of the form $(\mathbf{A}, \mathbf{A})$ are not very interesting from either the algebraic or the logical point of view. For a consistent logic $(\mathbf{A}, \mathbf{M})$, the concept of completeness has an elegant algebraic formulation: the logic is complete if and only if the ideal $\mathbf{M}$ is maximal in the algebra $\mathbf{A}$. ("Maximal ideal" in such contexts always means "maximal proper ideal.")

If $(\mathbf{A}, \mathbf{M})$ is a Boolean logic, it is natural to form the quotient system $\mathbf{A}/\mathbf{M}$. A necessary and sufficient condition that $\mathbf{A}/\mathbf{M}$ be a Boolean algebra is that it have at least two distinct elements, and a necessary and sufficient condition for that is exactly that $\mathbf{M}$ be a proper ideal in $\mathbf{A}$. On the other hand, a necessary and sufficient condition that $(\mathbf{A}, \mathbf{M})$ be complete is that $\mathbf{A}/\mathbf{M}$ have at most two distinct elements; if this is so, and if $\mathbf{M} \neq \mathbf{A}$, then $\mathbf{A}/\mathbf{M} = \mathbf{O}$. (Recall that $\mathbf{O} = \{0, 1\}$. On universal algebraic grounds, the condition that a proper ideal $\mathbf{M}$ be maximal in an algebra $\mathbf{A}$ is equivalent to the condition that the quotient algebra $\mathbf{A}/\mathbf{M}$ be *simple*, *i.e.*, that $\mathbf{A}/\mathbf{M}$ have no non-trivial proper ideals. The only simple Boolean algebra is $\mathbf{O}$.) Conclusion: a necessary and sufficient condition that a Boolean logic $(\mathbf{A}, \mathbf{M})$ be both consistent and complete is that $\mathbf{A}/\mathbf{M} = \mathbf{O}$.

For a deeper study of concepts such as consistency and completeness additional logical apparatus is needed; some of it is described below. First, a terminological warning. Consistency and completeness, as defined above, are usually called *simple consistency* and *simple completeness*; perhaps *syntactic* would be a more suggestive adjective here than *simple*. The point is that the propositional calculus, for instance, is usually considered to be a "language," and (simple) consistency and completeness are defined in terms of the intrinsic structure (syntax) of that language. Two related concepts, to be discussed below, are defined in terms of a possible external interpretation (meaning) of the language, and are therefore appropriately called *semantic*. The group-theoretic analog of a semantic concept is one that depends on constructions reaching outside the given group, *i.e.*, typically, on representations of the group. Thus, for instance, "characteristic subgroup" is a syntactic concept, while "character" is a semantic one.

In most of what follows it will be simpler to forget about Boolean logics and to consider Boolean algebras instead. The point is that if $(\mathbf{A}, \mathbf{M})$ is a consistent Boolean logic, then the quotient algebra $\mathbf{A}/\mathbf{M}$ can be formed and can be used to study virtually all the logically important properties of $(\mathbf{A}, \mathbf{M})$. Note that if a proposition in $\mathbf{A}$ is refutable, then its image in $\mathbf{A}/\mathbf{M}$ (its equivalence class modulo $\mathbf{M}$) is equal to 0, and, similarly, if a proposition in $\mathbf{A}$ is provable, then its image in $\mathbf{A}/\mathbf{M}$ is equal to 1. It is, accordingly, convenient to agree that if p is an element of a Boolean algebra, then "p is refutable" shall be a long way of saying "$p = 0$," and, similarly, "p is provable" shall be a long way of saying "$p = 1$."

9. Quantifiers. Within the framework of Boolean algebras (or logics) it is easy to give an algebraic formulation of the inference from the premises "Some Greeks are men" and "All men are mortal" to the conclusion "Some Greeks are men." This is not a misprint. Within the framework of Boolean algebras alone it is not possible to formulate the inference that allows, from the same premises, the conclusion "Some Greeks are mortal." The desired inference is justified not by manipulating with propositions as a whole, but by the intrinsic structure of its constituents, and, in particular, by a study of what "some" and "all" mean.

The clue is in the consideration of propositional functions. As their name indicates, propositional functions are functions whose values are propositions. If, for instance, for every natural number x, $p(x)$ is the sentence "x is even" and $q(x)$ is the sentence "$2x=1$," then p and q are propositional functions. (The present discussion is only heuristic, of course, but even so the contrasting use of "sentence" and "proposition" needs a little justification. The justification is a very common one in mathematics; it is convenient, though incorrect, to identify an equivalence class with a representative element. Recall that a proposition was defined above as an equivalence class of sentences. The solecism is the same as the one every analyst commits when he speaks of an element of L_2 as a function.) From the algebraic point of view, a propositional function is a function defined on an arbitrary non-empty set with values in a Boolean algebra.

To single out a particular theory for logical examination means, algebraically, to fix a Boolean algebra **B**. If, in addition, a certain non-empty set X is selected, then the ground is prepared for a discussion of propositional functions. (Intuitively the elements of X may be thought of as the objects that the propositions of the theory talk about.) The set **A** of all functions from X to **B** is in a natural way a Boolean algebra (pointwise operations), and the same is true of many of its subsets. The constant functions (*i.e.*, the functions p obtained by selecting an element p_0 in **B** and writing $p(x)=p_0$ for every x in X) constitute a subalgebra of the algebra **A**; that subalgebra is obviously isomorphic to **B**. In other words, the propositions of a theory are (or, rather, may be identified with) particular propositional functions of that theory.

What is the effect of "some" in a sentence such as "for some x, x is even"? (It is understood here that the underlying set X is the set of natural numbers; the value-algebra **B** has not been, and need not be, specified). The most striking effect it has is to convert a propositional function into a constant, or, *via* the identification convention of the preceding paragraph, into a constant function. This effect is analogous to the effect of "sup" in "$\sup_x f(x)$" and to the effect of "$\int_0^1 \cdots dx$" in "$\int_0^1 f(x)dx$" (where the function f is, say, a real-valued continuous function defined on the closed unit interval). Accordingly, the algebraic analogue of "some" ought to be a mapping that sends a certain Boolean algebra (propositional functions) into a certain subalgebra (constant functions). Such a mapping (usually denoted by $\exists$) is indeed at the basis of the theory of propositional functions.

It is now wise to abstract from the motivation. Consider a perfectly arbitrary Boolean algebra **A**, and let $\exists$ be a mapping of **A** into itself. If **A** were an algebra of propositional functions, and if $\exists$ were the operator "some," then, presumably, $\exists$ would satisfy some rather special conditions; the problem now is to find a reasonable list of conditions that characterize such a special $\exists$. This is easy; here are some of them:

(Q 1) $$\exists 0 = 0,$$

(Q 2) $$p \leqq \exists p,$$

(Q 3) $$\exists(p \vee q) = \exists p \vee \exists q,$$

(Q 4) $$\exists\exists p = \exists p.$$

(In these conditions, and also in (Q 5) below, p and q are arbitrary elements of **A**). The intuitive grounds for the conditions are easy to see. Suppose, for instance, that $q(x)$ is "$2x = 1$"; here, once more, **A** is to be thought of as an algebra of propositional functions whose domain is the set of natural numbers. Since in all reasonable theories of the arithmetic of natural numbers the sentence "$2x = 1$" is refutable, each value of the propositional function q is refutable, and therefore (in accordance with the agreement concerning the reduction of Boolean logics to Boolean algebras) the propositional function q is equal to the constant function 0. The same considerations apply to the sentence "for some x, $2x = 1$" and thus serve to complete the illustration of (Q 1). The remaining conditions (Q 2), (Q 3), and (Q 4) are illustrated similarly. In intuitive terms, (Q 2) says that each value of p implies that "for some x, $p(x)$"; the construction of analogous readings of (Q 3) and (Q 4) is left as an exercise to the reader.

It is amusing to observe that (except for notation) the conditions (Q 1)–(Q 4) are well-known to most mathematicians (but in a very different context). They are, all but verbatim, the Kuratowski axioms for a closure operator on a topological space. They do not, however, serve to characterize the "some" operator; in technical language, the theory of closure algebras is not co-extensive with the monadic functional calculus. Another glance at the conditions as they now stand should arouse the suspicion that something is missing; the trouble is that they relate $\exists$ to $\vee$ only (0 and $\leqq$ can be defined in terms of $\vee$), and say nothing about the relation of $\exists$ to either $\wedge$ or $'$. The missing condition is this:

(Q 5) $$\exists(\exists p)' = (\exists p)'.$$

In intuitive terms, (Q 5) serves as at least a partial reminder of the fact that the constant functions form a Boolean algebra (so that, in particular, they are closed under complementation), and that "some" applied to a constant function has no effect.

[Two remarks are in order. (1) The conditions (Q 1)–(Q 5) are equivalent to a shorter set, namely to (Q 1), (Q 2), and

(Q 6) $$\exists(p \wedge \exists q) = \exists p \wedge \exists q.$$

The proof of this equivalence is just axiom-chopping. The longer set was selected for presentation here because of its greater intuitive content. (2) The motivation of (Q 1)–(Q 5), and also of (Q 6), could have been, and sometimes is, based on quasi-geometric considerations, involving the formation of (possibly infinite) Boolean suprema. The didactic danger of this motivation is its emphasis on the intuitive idea that "some" is just an infinite "or." The idea is not wholly unsound, but real progress in algebraic logic was achieved only after the realization that $\exists$ is really a unary operation, not an infinitary one.]

The way is now clear to a precise definition: an *existential quantifier* is a mapping $\exists$ of a Boolean algebra into itself satisfying the conditions (Q 1)–(Q 5). Dually, a *universal quantifier* is a mapping $\forall$ of a Boolean algebra into itself satisfying the conditions

(Q′ 1) $$\forall 1 = 1,$$

(Q′ 2) $$\forall p = p,$$

(Q′ 3) $$\forall(p \wedge q) = \forall p \wedge \forall q,$$

(Q′ 4) $$\forall\forall p = \forall p,$$

(Q′ 5) $$\forall(\forall p)' = (\forall p)'.$$

It is easy to see that $\forall$ bears the same relation to the intuitive "all" as $\exists$ bears to "some."

There is a very close connection between existential quantifiers and universal ones: if $\exists$ is an existential quantifier on, say, a Boolean algebra $\mathbf{A}$, and if a mapping $\forall$ of $\mathbf{A}$ into itself is defined by

$$\forall p = (\exists p')',$$

then $\forall$ is a universal quantifier on $\mathbf{A}$; if, in reverse, a universal quantifier $\forall$ is given, and if $\exists$ is defined by

$$\exists p = (\forall p')',$$

then $\exists$ is an existential quantifier. ("Always p" is the same as "not sometimes not p," and "sometimes p" is the same as "not always not p"). The thoroughgoing symmetry of this situation justifies an asymmetric treatment; anything that can be said about an $\exists$ has an obvious dual about an $\forall$, and it is therefore sufficient to discuss in detail only one of these two objects. In what follows $\exists$ will be given preferred treatment, and, in fact, the word "quantifier" will be used from now on in the sense of "existential quantifier." (This is in line with the algebraists preference for ideals instead of filters; the dual of an algebraist would select $\forall$ for preferred treatment.) Universal quantifiers, whenever they must be considered, will always be given their full name.

10. Monadic algebras. Once the concept of a quantifier is at hand, it is immediately possible to generalize the concept of a Boolean algebra in a manner adapted to an algebraic answer to the question of why some Greeks are mortal. Technically the generalized algebras (called monadic algebras) play an inter-

mediate role; they are slightly more useful than Boolean algebras, but not nearly so useful as certain even more general algebras (the polyadic algebras that will be described a little later). Psychologically and historically, however, the intermediate generalization has some value. Its psychological value is that it exhibits in a simplified form some of the curious properties of its generalized version; its historical value is that it puts into a modern algebraic context the oldest known systematic treatment of logic, namely Aristotle's syllogistics.

A *monadic* (Boolean) *algebra* is a pair $(\mathbf{A}, \exists)$, where $\mathbf{A}$ is a Boolean algebra and $\exists$ is a quantifier on $\mathbf{A}$. (The word "monadic" serves as a reminder of the one additional operation that distinguishes monadic algebras from Boolean algebras.) The elementary theory of monadic algebras is a routine matter; subalgebras, homomorphisms, ideals, and similar universal algebraic concepts (with a qualifying "monadic" when clarity demands it) are defined in a completely unsurprising manner. Free monadic algebras can be defined and studied (if desired) by the techniques used in the study of free Boolean algebras; the analogue of the propositional calculus is called the *monadic functional calculus*.

The concept of a monadic logic arises naturally in connection with the theory of provability and refutability in the monadic functional calculus: a *monadic logic* is a pair $(\mathbf{A}, \mathbf{M})$, where $\mathbf{A}$ is a monadic algebra (with quantifier $\exists$, say) and $\mathbf{M}$ is a monadic ideal in $\mathbf{A}$. (It is convenient here, as in other parts of algebra, to be mildly forgetful of the completely rigorous definitions, and, accordingly, to identify a monadic algebra with the underlying Boolean algebra. If this were not done, a monadic logic would have to be denoted by a symbol such as $((\mathbf{A}, \exists), \mathbf{M})$.) The fact that a monadic ideal is a Boolean ideal invariant under the application of $\exists$ corresponds to the logical fact that if p is a refutable propositional function, then $\exists p$ is also refutable.

To discuss the (simple, or syntactic) consistency and completeness of Boolean logics, it is desirable to introduce a new term: an element p of a monadic algebra will be called *closed* if $\exists p = p$. Intuitively, closed propositions correspond to what were called constant functions before, or, in other words, to propositions instead of propositional functions. There is a natural way of associating a Boolean logic $(\mathbf{A}_0, \mathbf{M}_0)$ with every monadic logic $(\mathbf{A}, \mathbf{M})$; the algebra $\mathbf{A}_0$ is the set of all closed elements of $\mathbf{A}$ and the ideal $\mathbf{M}_0$ is the intersection of $\mathbf{M}$ with $\mathbf{A}_0$. A monadic logic $(\mathbf{A}, \mathbf{M})$ is called syntactically consistent (or complete) if the associated Boolean logic $(\mathbf{A}_0, \mathbf{M}_0)$ is consistent (or complete).

Why is it necessary to modify the Boolean definitions of consistency and of completeness for the monadic situation? Consistency could conceivably be defined by the requirement that for no p in $\mathbf{A}$ should both p and p' belong to $\mathbf{M}$, and, similarly, completeness could be defined by the requirement that for every p in $\mathbf{A}$ either p or p' should belong to $\mathbf{M}$. What is wrong with these definitions? The answer is that there is nothing wrong as far as consistency is concerned, and very much is wrong as far as completeness is concerned. The alternative definition of consistency is equivalent to the one officially adopted above, but the alternative definition of completeness is out of harmony with both the official

definition and common sense. If, as in some previous examples, $p(x)$ is "x is even," then it is not at all reasonable to demand of a logic sufficient power to settle the provability or refutability of the propositional function p. The sentences "for some x, x is even" and "for all x, x is even" are closed; a reasonable logical theory of arithmetic should declare each of them to be either provable or refutable. The function p, however, is not a sentence; common sense demands that both it and its negation should fail to be either provable or refutable.

11. Syllogisms. The discussion of monadic algebras and logics up to this point was merely an adaptation of the corresponding discussion of Boolean facts. Some progress has been made, nevertheless; monadic logics (unlike Boolean logics) contain the general theory of syllogisms. For an example, consider again the premises "Some Greeks are men" and "All men are mortal," and consider the desired conclusion "Some Greeks are mortal." To make an algebraic model of the situation, let X be an appropriate set and let $\mathbf{B}$ be an appropriate Boolean algebra of propositions about the elements of X. The set X, for instance, could be the set of all animals, and $\mathbf{B}$ could be an algebra containing, for each x in X, the propositions "x is Greek," "x is a man," and "x is mortal." (This description is, of course, much too colloquial for complete precision, but there is no difficulty at all in converting it into honest mathematics.) If these propositions are denoted by $p(x)$, $q(x)$, and $r(x)$, respectively, and if $\mathbf{A}$ is the algebra of all propositional functions such as p, q, and r, then $\mathbf{A}$ (with the intuitively obvious "some" quantifier in the role of $\exists$) is a monadic algebra suitable for the study of the inference described above. The first premise is $\exists(p \wedge q)$, the second premise is $\forall(q \rightarrow r)$, and the conclusion is $\exists(p \wedge r)$. (The universal quantifier $\forall$ here is the natural dual of the given existential quantifier $\exists$.) The algebraic justification of the inference is that if $\mathbf{A}$ is the monadic algebra of a monadic logic such that both premises belong to the filter of provable propositions, then the conclusion also belongs to that filter.

There is a special aspect of the theory of syllogisms that still remains to be converted into algebra; it is exemplified by the classical premises "Socrates is a man" and "All men are mortal," together with the conclusion "Socrates is mortal." In highly informal language, the trouble with this syllogism is that the algebraic theory (so far) is equipped to deal with generalities only, and is unable to say anything concrete; Socrates, however, is a concrete individual entity. The preceding paragraph shows that a monadic algebra can be taught to say "All men are mortal." The reason is that "manhood" can easily be thought of as an element of a monadic algebra, since it is the obvious abstraction of the propositional function whose value at each point x of some set is "x is a man." Socrates, on the other hand, is a "constant," and there is no immediately apparent way of pointing to him. (The use of the word "constant" here and below is quite different from its earlier use in the phrase "constant function." The important concept now is what logicians call an individual constant.) A classical artifice, designed to deal with just this difficulty, is to promote Socrates to a propositional function, *i.e.*, to identify him with the function whose value at x is

"x is Socrates." This procedure is both intuitively and algebraically artificial; Socrates should not be a propositional function but a possible argument of such functions.

To find the proper algebraization of the concept of a constant it is necessary only to recall that the elements of a monadic algebra are abstractions of the concept of a propositional function, and to determine the algebraic effect of replacing the argument of such a function by some fixed element of its domain. If p is a propositional function with domain X, and if $x_0 \in X$, then $p(x_0)$ is a proposition, or, *via* the obvious identification convention, a propositional function with only one value. The mapping $p \to p(x_0)$ (the evaluation map induced by x_0) clearly preserves the Boolean operations (*i.e.*, suprema, infima, and complements). If the function p itself has only one value (equivalently, if $p = \exists q$ for some q), then that value is $p(x_0)$, *i.e.*, the mapping leaves the range of $\exists$ elementwise fixed. If, on the other hand, $\exists$ is applied to the (constant) function $p(x_0)$, the result is the same function, *i.e.*, $\exists$ leaves the range of the mapping elementwise fixed. These considerations motivate the following general definition: a *constant* of a monadic algebra $\mathbf{A}$ is a Boolean endomorphism c of $\mathbf{A}$ such that

$$c\exists = \exists \quad \text{and} \quad \exists c = c.$$

This definition is applicable to the mortality of Socrates. If, as before, q is manhood and r is mortality, and if Socrates is taken to be a constant, say c, of a monadic algebra containing q and r, then the algebraic justification of the classical syllogism is this: if $\mathbf{A}$ is the monadic algebra of a monadic logic such that both $\forall(q \to r)$ and cq belong to the filter of provable propositions, then cr also belongs to that filter.

Constants are much more important than their more or less casual introduction above might indicate; the concept of a constant (suitably generalized to the polyadic situation) is probably the most important single concept in algebraic logic. This should not be too surprising; in the intuitive interpretation, the constants of a theory constitute, after all, the subject matter that the propositions of the theory talk about. Algebraically constants play a crucial role in the representation theory of monadic (and polyadic) algebras.

12. Quantifier algebras. The theory of propositional functions of only one variable (and of their abstract algebraic counterparts) is as insufficient for the understanding of logic and its applications as the theory of ordinary numerical functions of one variable is insufficient for the understanding of calculus. Mathematics contains not only sentences of the form "x is positive," but also sentences such as "x is less than y" and "x is between y and z." Although a mathematician with no training in modern logic is likely to be suspicious when he is told that syllogisms are not enough for mathematics, the basis of that assertion is nothing more profound or esoteric than the need to consider propositional functions of several variables. This is all that DeMorgan meant when he said that the scholastics, after two millennia of Aristotelean tradition, were still unable to prove that if a horse is an animal, then a horse's tail is an animal's tail.

Suppose, to begin modestly, that one wants to study propositional functions of three variables. The most immediately apparent new phenomenon is the possibility of partial quantification: the theory must be able to treat sentences of the form "there is an x_1 and there is an x_3, such that $p(x_1, x_2, x_3)$." (Here, as before, it is enough to discuss existential quantifiers; the corresponding theory of universal quantifiers is obtained by a simple dualization.) Another way of expressing the phenomenon is to say that the appropriate algebraic system must possess not one quantifier but many. What appears to be going on is this: if I is the set of relevant indices, so that $I = \{1, 2, 3\}$ in the present example, then there is a quantifier (*i.e.*, an existential quantifier) corresponding to each subset (*e.g.*, to $\{1, 3\}$) of the set I. If $\exists(J)$ is the quantifier corresponding to the subset J of I, then, of course, the way that $\exists(J)$ depends on J should be specified. The example shows the way. If J is empty, then prefixing $\exists(J)$ to p should produce no change in p, and if both J and K are subsets of I, then prefixing $\exists(J)$ and $\exists(K)$ to p, in either order, should have the same effect as prefixing the quantifier corresponding to the union of J and K.

The preceding paragraph suggests the definition of an algebraic system, which, however, turns out to fall far short of what is needed. It is worth a brief look anyway. Call a *quantifier algebra* a triple $(\mathbf{A}, I, \exists)$, where $\mathbf{A}$ is a Boolean algebra, I is a set, and $\exists$ is a function from subsets of I to quantifiers on $\mathbf{A}$ such that

$$(\exists\ 1) \qquad \exists(\varnothing)p = p$$

whenever $p \in \mathbf{A}$, and

$$(\exists\ 2) \qquad \exists(J)\exists(K) = \exists(J \cup K)$$

whenever J and K are subsets of I. In analogy with logical usage, an element of the set I will be called a *variable* (or, in more detail, an *individual variable*) of the quantifier algebra $(\mathbf{A}, I, \exists)$. Caution: a variable, in this sense, does not vary at all; it merely serves as a reminder of the place into which a "variable," in the intuitive sense, could be substituted, if, that is, the elements of $\mathbf{A}$ were propositional functions and not elements of a quite abstract Boolean algebra.

The concept of a quantifier algebra is a proper generalization of the concept of a monadic algebra; a quantifier algebra $(\mathbf{A}, I, \exists)$ for which the set I of variables consists of exactly one element may be identified with the monadic algebra $(\mathbf{A}, \exists(I))$. The *degree* of a quantifier algebra is defined to be the cardinal number of the set of its variables; the last comment says that quantifier algebras of degree 1 are essentially the same as monadic algebras. It is crucial for applications to permit the consideration of quantifier algebras of infinite degree. While, to be sure, each particular propositional function that is likely to arise has only finitely many arguments, the number of arguments in a reasonably extensive theory (*e.g.*, in mathematics) is not likely to be bounded. To keep the generalization from running away with itself, it is often advisable to restrict the study of quantifier algebras to the locally finite case. The quantifier algebra $(\mathbf{A}, I, \exists)$ is

called *locally finite* if to every element p of $\mathbf{A}$ there corresponds a finite subset J of I such that $\exists(I-J)p=p$; this is the abstract formulation of the concrete promise that no propositional function under consideration will actually depend on infinitely many variables.

13. Polyadic algebras. The reason that quantifier algebras are an inefficient logical tool is that they do not allow the discussion of transformations of variables. Consider, for instance, a sentence of the form "if $p(x_1, x_2)$ and $p(x_2, x_1)$, then $x_1=x_2$." Such sentences usually occur in the definition of a partial order; they indicate the need for discussing the passage from $p(x_1, x_2)$ to $p(x_2, x_1)$. For a more complicated example of the same type consider the problem of converting $p(x_1, x_2, x_3, x_4)$ into $p(x_4, x_4, x_1, x_3)$. There is no way of describing such conversions in terms of Boolean operations and quantifications alone; the best way of incorporating them into the general theory is by postulating them outright. What appears to be going on is this: to every transformation τ on I (*i.e.*, to every mapping of the set I into itself) there corresponds a mapping, say $\mathbf{S}(\tau)$, of the set of propositional functions under consideration into itself. The mappings $\mathbf{S}(\tau)$ preserve the Boolean operations (*i.e.*, they are Boolean endomorphisms). If τ is the identity transformation on I (to be denoted by δ), then $\mathbf{S}(\tau)$ leaves every p invariant; if both σ and τ are transformations on I, then the effect of $\mathbf{S}(\sigma)$ on $\mathbf{S}(\tau)p$ is the same as the effect of $\mathbf{S}(\sigma\tau)$ on p.

In analogy with quantifier algebras, it is possible to define a *transformation algebra* as a triple $(\mathbf{A}, I, \mathbf{S})$, where $\mathbf{A}$ is a Boolean algebra, I is a set, and $\mathbf{S}$ is a function from transformations on I to Boolean endomorphisms on $\mathbf{A}$, such that

$$(\mathbf{S}\ 1) \qquad \mathbf{S}(\delta)p = p$$

whenever $p \in \mathbf{A}$, and

$$(\mathbf{S}\ 2) \qquad \mathbf{S}(\sigma)\mathbf{S}(\tau) = \mathbf{S}(\sigma\tau)$$

whenever σ and τ are transformations on I.

Neither quantifier algebras nor transformation algebras are of much logical significance; the important concept is one that possesses both structures at the same time. It is a matter of universal algebraic experience, however, that it is not enough to impose two different structures on the same set; in order to get a usable theory, it is necessary to describe the compatibility conditions that relate the two structures to each other. (A topological group is not merely a set that is simultaneously a group and a topological space.) The appropriate conditions are discovered by experimentation with the special structures that the theory is intended to generalize; their final justification is their success. To examine here the experimentation that led to the definition of polyadic algebras would be more boring than profitable; suffice it to report the results. A *polyadic* (Boolean) *algebra* is a quadruple $(\mathbf{A}, I, \mathbf{S}, \exists)$ subject to the following conditions: (i) $(\exists\ 1)$ and $(\exists\ 2)$ hold, or, more precisely, $(\mathbf{A}, I, \exists)$ is a quantifier algebra, (ii) $(\mathbf{S}\ 1)$ and $(\mathbf{S}\ 2)$ hold, or, more precisely, $(\mathbf{A}, I, \mathbf{S})$ is a transformation algebra, and (iii)

(∃S 1) $$\mathbf{S}(\sigma)\exists(J) = \mathbf{S}(\tau)\exists(J)$$

whenever J is a subset of I and σ and τ are transformations on I such that $\sigma i = \tau i$ for all i in $I - J$, and

(∃S 2) $$\exists(J)\mathbf{S}(\tau) = \mathbf{S}(\tau)\exists(\tau^{-1}J)$$

whenever J is a subset of I and τ is a transformation on I such that two distinct elements of I are never mapped by τ onto the same element of J.

The conditions (∃S 1) and (∃S 2) were recorded here for the sake of completeness only; no technical use will be made of them in this merely descriptive report. It is perhaps worth remarking though that they (or, rather, the provisos that accompany them) are not so complicated as they seem on first sight. What they amount to is a condensation of the usual and intuitively obvious relations between quantifications and transformations. Suppose, for example, that i and j are distinct elements of I; let J be the singleton $\{i\}$ and let τ be the transformation that maps i onto j and everything else (including j itself) onto itself. Since τ agrees with δ outside J, it follows from (∃S 1) that $\mathbf{S}(\tau)\exists(J) = \exists(J)$; this equation corresponds to the familiar fact that once a variable has been quantified, the replacement of that variable by another one has no further effect. To get another example, note, for the same τ and J, that $\tau^{-1}J = \varnothing$. It follows from (∃S 2) that $\exists(J)\mathbf{S}(\tau) = \mathbf{S}(\tau)$; this equation corresponds to the familiar fact that once a variable has been replaced by another one, a quantification on the replaced variable has no further effect.

[Polyadic algebras stand in the same relation to the so-called *pure* first-order functional calculus as do Boolean algebras to the propositional calculus. The simplest example of an *applied* functional calculus is one that is equipped to discuss the concept of equality. By a suitable adaptation of standard logical methods, such applied calculi can be treated within the framework of polyadic algebras; the details are of no relevance here. What should be mentioned, however, is that there is another way of algebraizing the functional calculus with equality, namely, *via* Tarski's concept of a cylindric algebra. Roughly speaking, a cylindric algebra is a quantifier algebra together with certain distinguished elements; the distinguished elements play the role of sentences that assert equations among variables. The theory of cylindric algebras is an efficient algebraic tool for studying calculi with equality. The exact relations between polyadic algebras and cylindric algebras are of considerable technical interest; they are still in the process of being clarified. It is already known, however, that in most important cases the two concepts are equivalent. The way that transformations enter into the theory of cylindric algebras is an ingenious trick; the idea is that to say "$p(x_1, x_1)$" is the same as to say "there is an x_2 such that $p(x_1, x_2)$ and $x_2 = x_1$."]

14. Semantic concepts. The discussion of polyadic logics and their syntactic consistency and completeness proceeds in a straightforward manner. A *polyadic logic* is a pair $(\mathbf{A}, \mathbf{M})$, where $\mathbf{A}$ is a polyadic algebra and $\mathbf{M}$ is a polyadic ideal in

A. An element p of a polyadic algebra **A** is called *closed* if $\exists(I)p=p$ (where I, of course, is the set of variables of **A**). Just as for monadic logics, there is a natural way of associating a Boolean logic $(\mathbf{A}_0, \mathbf{M}_0)$ with every polyadic logic $(\mathbf{A}, \mathbf{M})$; the algebra $\mathbf{A}_0$ is the set of all closed elements of **A** and the ideal $\mathbf{M}_0$ is the intersection of **M** with $\mathbf{A}_0$. A polyadic logic $(\mathbf{A}, \mathbf{M})$ is called syntactically consistent (or complete) if the associated Boolean logic $(\mathbf{A}_0, \mathbf{M}_0)$ is consistent (or complete).

All this is pretty routine stuff by now. The element of novelty is in the study of the "semantic" theory of polyadic logics, *i.e.*, in the study of "interpretations" of a logic in a "model" and in the study of "truth" and "validity." (It must be admitted that these concepts could have been introduced in connection with Boolean logics and monadic logics also. Since, however, in those simple situations, they exhibit the deceptive simplicity of a degenerate case, their premature study would have been more confusing than helpful.) The natural habitat of semantic theories is situated somewhere near a polyadic *logic*, but the machinery for examining such a theory is easier for a polyadic *algebra*. The reduction of (syntactically consistent) logics to algebras is an easy step: if $\mathbf{M}\neq\mathbf{A}$, then $\mathbf{A}/\mathbf{M}$ can be formed and can be used to study $(\mathbf{A}, \mathbf{M})$. (This kind of thing was done before; *cf.* the corresponding discussion in the Boolean case.) Some concepts disappear in the passage from $(\mathbf{A}, \mathbf{M})$ to $\mathbf{A}/\mathbf{M}$ and others take on an interesting new guise. Thus, for instance, to say that a polyadic algebra is (syntactically) consistent is to say merely that it is there; on the other hand, to say that a polyadic algebra is (syntactically) complete is to say that it is simple. (Just as for Boolean logics, it turns out that a polyadic logic $(\mathbf{A}, \mathbf{M})$, with $\mathbf{M}\neq\mathbf{A}$, is syntactically complete if and only if the polyadic ideal **M** is a maximal ideal in the polyadic algebra **A**.)

The usual way to describe a model is this: take a non-empty set X, interpret the variables (*i.e.*, the elements of I) as variables (in the intuitive sense) varying over X, and interpret the elements of **A** as propositional functions on an appropriate Cartesian power of X with values that are either true or false. This is somewhat vague; a slight change in language converts it, however, into an algebraically precise and clean definition. If X is a non-empty set and if I is an arbitrary set, the set of all functions from the Cartesian product X^I into the Boolean algebra **O** possesses in a natural way the structure of a polyadic Boolean algebra; a *model* is, by definition, a polyadic subalgebra of such a total functional algebra.

Once the concept of a model is known, the definitions of the remaining semantic concepts are automatic. The purpose of the following paragraphs is first to define the usual semantic concepts and then to examine their algebraic significance.

An *interpretation* of a polyadic algebra $(\mathbf{A}, I, \mathbf{S}, \exists)$ in a model is a polyadic homomorphism from **A** onto the model. An element p of **A** is *true* in an interpretation f if $fp=1$; if $fp=0$, then p is *false* in the interpretation. (Caution: fp is a function from X^I into **O** and is therefore not necessarily either 0 or 1. To say that p is true for f means that the image fp of p under f is the constant function

whose value is 1.) An element p of **A** is *valid* if it is true in every interpretation; if p is false in every interpretation, then p is called *contravalid*. An element p of **A** is *satisfiable* if there is an interpretation for which it is not false (*i.e.*, if p is not contravalid).

The algebra **A** is *semantically consistent* if it has an interpretation in a model. The definition of semantic completeness requires a little more motivation. It is a trivial consequence of the definition of Boolean homomorphism (sometimes even built in as part of the definition) that the unit element of **A** is always valid, and, dually, that the zero element is always contravalid; in logical terms this says that provable propositions are valid, and refutable ones are contravalid. (The assumption of syntactic consistency is usually, and quite properly, made explicit at this point. In the present discussion, dealing with algebras instead of logics, the role of that assumption is played by the fact that **A** is an algebra at all, or, in more detail, that **A** has a unit element distinct from zero.) Semantic completeness requires the converse assertions; the algebra **A** is *semantically complete* if the unit is the only valid element, or, equivalently, if every non-zero element is satisfiable. In logical terms these requirements say that everything valid is provable, or, equivalently, that everything that is not refutable is satisfiable.

All the preceding definitions are based on one of them, namely, on the definition of a model. The situation can be better understood if it is first generalized and then specialized in a new direction. A model was defined as an element of a certain particular class of polyadic algebras, but, in the subsequent definitions, no special properties of that class were used. Those definitions, in other words, are relative to the prescribed class $\mathfrak{M}$ of models, and to nothing else; a significant generalization of them is obtained if $\mathfrak{M}$ is replaced by an arbitrary class $\mathfrak{C}$ of algebras. The concepts so obtained may be referred to by using $\mathfrak{C}$ as a prefix, so that expressions such as "$\mathfrak{C}$-interpretation" and "$\mathfrak{C}$-valid" now make sense. (Only in case $\mathfrak{C}=\mathfrak{M}$ may the prefix be omitted.) An interesting specialization is obtained by letting the class $\mathfrak{S}$ of all simple algebras play the role of $\mathfrak{C}$. The discussion that follows will be in terms of $\mathfrak{S}$ rather than $\mathfrak{M}$; the vitally important relation between $\mathfrak{S}$ and $\mathfrak{M}$ will be treated afterward.

To say that an algebra **A** is semantically $\mathfrak{S}$-consistent means, by definition, that there exists a homomorphism from **A** onto a simple algebra. This, in turn, is equivalent, on universal algebraic grounds, to the existence of a maximal ideal in **A**. The problem of the existence of maximal ideals is well known in algebra; the cases in which the solution is affirmative can usually be settled by a straightforward application of Zorn's lemma. The theory of polyadic algebras is one of these pleasant cases; Zorn's lemma applies and proves that maximal ideals always do exist. In other words, every polyadic algebra is (semantically) $\mathfrak{S}$-consistent. The situation is reminiscent of syntactic consistency, and, in fact, it turns out that the two concepts are the same. It is profitable to return (but just for a moment) to logics instead of algebras. Recall that a logic (**A**, **M**) is syntactically consistent if and only if **M** is a proper ideal in **A**. The general definition

of semantic $\mathfrak{C}$-consistency for a logic $(\mathbf{A}, \mathbf{M})$ requires the existence of an algebra $\mathbf{C}$ in the class $\mathfrak{C}$ and the existence of a homomorphism f from $\mathbf{A}$ onto $\mathbf{C}$ such that $fp = 0$ whenever $p \in \mathbf{M}$. If $\mathfrak{C}$ is $\mathfrak{S}$, then this is equivalent to the requirement that $\mathbf{M}$ be included in some maximal ideal. Since Zorn's lemma can be used to show not only that maximal ideals exist, but that, in fact, every proper ideal is included in some maximal ideal, syntactic consistency and semantic $\mathfrak{S}$-consistency are the same whether they are approached *via* logics ($\mathbf{M}$ is a proper ideal) or *via* algebras ($\mathbf{A}$ is a non-degenerate algebra).

To say that an algebra $\mathbf{A}$ is semantically $\mathfrak{C}$-complete means, by definition, that for every non-zero element p of $\mathbf{A}$ there exists an algebra $\mathbf{C}$ in the class $\mathfrak{C}$ and there exists a homomorphism f from $\mathbf{A}$ onto $\mathbf{C}$ such that $fp \neq 0$. In this formulation $\mathfrak{C}$-completeness may not look familiar even to a professional algebraist; an easy argument serves, however, to prove that the definition is equivalent to one that is very well known indeed. The fact is that $\mathbf{A}$ is $\mathfrak{C}$-complete if and only if $\mathbf{A}$ is a *subdirect sum* of algebras belonging to the class $\mathfrak{C}$. The argument (and even the definition of subdirect sum) will be omitted here; they are mentioned only to show that the concept ($\mathfrak{C}$-completeness for an arbitrary $\mathfrak{C}$) does fit into the framework of algebra.

In view of the relation between maximal ideals on the one hand and homomorphisms onto simple algebras on the other hand, the definition of semantic $\mathfrak{S}$-completeness for an algebra $\mathbf{A}$ reduces to this: to every non-zero element of $\mathbf{A}$ there corresponds a maximal ideal not containing it. Equivalently: the intersection of all the maximal ideals of $\mathbf{A}$ is the singleton $\{0\}$. This too is a well-known algebraic phenomenon. In analogy with some other parts of algebra, a polyadic algebra will be called *semisimple* if the intersection of all its maximal ideals is trivial; the result is that a polyadic algebra is $\mathfrak{S}$-complete if and only if it is semisimple, or, equivalently, if and only if it is a subdirect sum of simple algebras.

15. The Gödel theorems. The title of this paper promised a discussion of the basic concepts of algebraic logic, and indeed the definitions of the basic *concepts* were emphasized much more than the basic *results* concerning them. As a partial rectification of this unbalanced state of affairs the present (concluding) section is devoted to the statement of two of the deepest theorems of the field. The original formulations and proofs of these justly celebrated theorems are due to Gödel; the results are known as the Gödel completeness theorem and the Gödel incompleteness theorem.

The word "completeness" in "completeness theorem" refers to semantic completeness. The algebraic statement of the theorem is short and elegant: *every locally finite polyadic algebra of infinite degree is semantically complete* (or, equivalently, $\mathfrak{M}$-complete). The crux of the proof is a characterization of simple polyadic algebras. It is very easy to see that every model is a simple algebra; the hard thing to prove is that, conversely, every simple, locally finite polyadic algebra of infinite degree is (isomorphic to) a model. The proof makes use of the polyadic generalization of what was called a "constant" of a monadic algebra.

In terms of the symbols used above, the easy fact is that $\mathfrak{M} \subset \mathfrak{S}$ and the hard one is that (in the presence of local finiteness and infinite degree) $\mathfrak{S} \subset \mathfrak{M}$. In view of the equation $\mathfrak{S} = \mathfrak{M}$, semantic completeness (*i.e.*, $\mathfrak{M}$-completeness) is the same as $\mathfrak{S}$-completeness, and, consequently, the Gödel completeness theorem implies that every locally finite polyadic algebra of infinite degree is semisimple. In fact, ***every*** polyadic algebra is semisimple, and this assertion is sometimes taken to be the algebraic version of the Gödel completeness theorem. The proof of semisimplicity is quite easy; the crucial fact is not that something is semisimple but that in the most important cases semisimplicity is the same as semantic completeness.

The word "incompleteness" in "incompleteness theorem" refers to syntactic incompleteness. The result here is not so general as the completeness theorem; the subject matter is not all polyadic algebras, but only relatively few of them. The algebras covered by the theorem are usually described by saying that they are adequate for elementary arithmetic. Because a precise explanation of what this means would involve a rather long and technical detour, no such explanation will be presented here, and, consequently, the incompleteness theorem will be described rather than stated. The striking qualities of the theorem are sufficiently great to remain visible even under such cavalier treatment.

Even if the definition of the class of algebras under consideration is not made explicit, it is convenient to have a short phrase in which to refer to them; in what follows "Peano algebra" will be used instead of "polyadic algebra that is adequate for elementary arithmetic." The Gödel incompleteness theorem asserts the existence of "undecidable propositions." Since in the passage from logics to algebras the statement that an element p is refutable (or provable) was identified with the statement that $p = 0$ (or $p = 1$), and since "undecidable" means "neither refutable nor provable," the assertion reduces to the existence of a (closed) element different from both 0 and 1. The ideal generated by such an element is a non-trivial proper ideal. Conversely, every non-trivial proper ideal contains an undecidable (closed) element. These facts indicate that the Gödel incompleteness theorem asserts the existence, in Peano algebras, of non-trivial proper ideals; the definition of syntactic completeness shows now that the theorem does indeed assert that something is (syntactically) incomplete. The usual, logical, formulation explicitly makes the assumption that the underlying logic is consistent; the treatment of algebras instead of logics makes it unnecessary to mention such an assumption here.

The Gödel theorem does not assert that every Peano algebra is syntactically incomplete. It asserts, instead, that the definition of Peano algebras is not a faithful algebraic transcription of all intuitive facts about elementary arithmetic. In algebraic terms this means that while some Peano algebras may be syntactically complete, there definitely exist others that are not. The situation is analogous to the one in the theory of groups. A class of polyadic algebras could be defined that is adequate for a discussion of elementary group theory. For this class the assertion that every two elements of a group commute would be un-

decidable, in the obvious sense that the assertion is true for some groups (and, therefore, corresponds to the unit element of some of the polyadic algebras under consideration) and false for others.

What has been said so far makes the Gödel incompleteness theorem take the following form: not every Peano algebra is syntactically complete. In view of the algebraic characterization of syntactic completeness this can be rephrased thus: not every Peano algebra is a simple polyadic algebra. This is the description that was promised above. What follows is another rephrasing of this description; the rephrasing, possibly of some mnemonic value, makes its point by making a pun. Consider one of the systems of axiomatic set theory that is commonly accepted as a foundation for all extant mathematics. There is no difficulty in constructing polyadic algebras with sufficiently rich structure to mirror that axiomatic system in all detail. Since set theory is, in particular, an adequate foundation for elementary arithmetic, each such algebra is a Peano algebra. The elements of such a Peano algebra correspond in a natural way to the propositions considered in mathematics; it is stretching a point, but not very far, to identify such an algebra with mathematics itself. Some of these "mathematics" may turn out to possess no non-trivial proper ideals, *i.e.*, to be syntactically complete; the Gödel theorem implies that some of them will certainly be syntactically incomplete. The conclusion is that the crowning glory of modern logic (algebraic or not) is the assertion: ***mathematics is not necessarily simple.***

ZERO INTEGRAL HOMOGENEOUS SYMMETRIC FUNCTIONS IN CERTAIN RINGS*

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In this paper†, we shall discuss a type of function which we shall define at the start. If $f(c_0, c_1, \cdots, c_n)$ is a function with the c's in a ring, then f is called an *integral homogeneous symmetric function of degree n*, written IHSF, of degree n provided it satisfies:

(i) f is a polynomial in the c's which is unchanged by any permutation of the c's.

(ii) $f(tc_0, tc_1, \cdots, tc_n) = t^n f(c_0, c_1, \cdots, c_n)$.

Each elementary symmetric function is an IHSF. It is well known that in the field of real numbers each elementary symmetric function of the nth roots of unity is zero except the one of highest degree. More generally if

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

has a decomposition in a field, then in this field, $a_i = S_i(-1)^i$, where S_i is the

* This paper was supported in part by National Science Foundation Grant G1397.

† The author wishes to thank the referee for valuable suggestions relative to this paper.

sum of the products of the roots of the above equation, taken i at a time, $i=1, 2, \cdots, n$.

Many theorems are known which give us congruences involving symmetric functions on the residue classes prime to a modulus. In some of these theorems the modulus is prime; in others, it is a power of a prime; and in still others, the modulus may have several distinct prime divisors. Some theorems have been given which relate symmetric functions with Bernoulli numbers [3, 7, 9]. Several theorems [1, 3, 9, 10] are also known which involve congruence relations on symmetric functions of residue classes not necessarily prime to a composite modulus. Symmetric functions have often been studied as products of the elementary symmetric functions; however, P. Hensel [6] studied the class of integral homogeneous symmetric functions and using the property of homogeneity, as we do in the proof of our main theorem, equation (2), obtained easily:

An IHSF of the numbers of the set A: $1, 2, \cdots, p-1$, *for* p *a prime, such that the degree of the function is not a multiple of* $p-1$, *is divisible by* p.

LaGrange's identity yields as a special case of Hensel's theorem, the well-known theorem:

If S_i *is the sum of the products of the elements of the set* A *above, taken* i *at a time,* $0<i<p-1$, *then* $S_i \equiv 0 \pmod{p}$.

Authors of recent textbooks have neglected both the method of Hensel and his theorem, but mention this special case. Wolstenholme's theorem is also often mentioned:

If t *varies over the set* A *and* $1/t$ *is defined by* $(1/t)t \equiv 1 \pmod{p^2}$, *then* $\sum 1/t \equiv 0 \pmod{p^2}$, *for* $p>3$.

It is the object of this paper to prove a theorem which contains as a special case a corollary which contains Hensel's theorem and one similar to that of Wolstenholme. Both the methods used and the results obtained are similar to methods and results of Vandiver's paper [9]. This together with the fact that our theorem also applies to symmetric functions involving non-units as well as units in certain rings suggests its considerable generality. The statement that u is a *unit* of the ring R means that R contains a multiplicative identity e, and that u divides this identity; e is called the *unity* element.

In a recent paper [11] the writer discussed cosets and divisor groups in semi-groups. If a commutative semi-group S contains a subsemi-group S' and a is an element of S, the *coset* aS' was defined to be the set of elements obtained by multiplying a by each element of S'; a was called the *coset multiplier* of aS'. A subgroup G of S was called a *divisor group* if: (i) Each element of S is in some coset aG; (ii) The cosets of S with respect to G are disjoint; and (iii) This division into cosets with respect to G is unique in the sense that each two divisions of S into cosets with respect to G contain the same cosets. For instance the group of units modulo 6 is a divisor group of the multiplicative semi-group modulo

6. A minimal set M of coset multipliers such that each element of S can be written in the form $a_i g$ where a_i is an element of M and g is an element of G will be called, in the present paper, a *complete set of coset multipliers of S with respect to G*. We shall need Theorem 7 and a special case of Theorem 1 of [11]. We state them respectively below as theorems B and A.

THEOREM A. *A finite semi-group S contains the subgroup G as a divisor group if the identity of G is an identity of S.*

THEOREM B. *If a semi-group is divided into left cosets with respect to a left divisor group G, a necessary and sufficient condition that two left cosets aG and bG be identical is that there exist an element g of G such that $a=bg$.*

We now proceed with our principal theorem.

THEOREM. *Let a finite commutative ring R with unity e contain among its units a multiplicative divisor group G of order a and an element i such that $i^{ad}-e$ is also a unit for d a positive integer; then each IHSF of degree d of the a-th powers of a complete set of coset multipliers of R with respect to G is zero in R.*

Proof. By using Theorem A we can write the distinct elements of R as

$$
\begin{aligned}
&w_1,\ w_1g_{s_1},\ w_1g_{s_2},\ \cdots,\ w_1g_{s_x},\\
&w_2,\ w_2g_{t_1},\ w_2g_{t_2},\ \cdots,\ w_2g_{t_y},\\
&\cdots\cdots\cdots\cdots\\
&w_b,\ w_bg_{u_1},\ w_bg_{u_2},\ \cdots,\ w_bg_{u_z},
\end{aligned}
\tag{1}
$$

where the g's are in G, and the w's form a complete set of coset multipliers of R with respect to G. Let S be an IHSF of degree d of the a-th powers of the w's. Then

$$
S((iw_1)^a, (iw_2)^a, \cdots, (iw_b)^a) = i^{ad}S(w_1^a, w_2^a, \cdots, w_b^a). \tag{2}
$$

If $1 \leqq k \leqq b$, we have from (1) that there exists an integer m, $1 \leqq m \leqq b$, and an element g_r of G such that

$$
iw_k = w_mg_r. \tag{3}
$$

Hence, since R is commutative

$$
(iw_k)^a = (w_mg_r)^a = w_m^a. \tag{4}
$$

Now as w_k varies over the complete set of coset multipliers in (3), so also does w_m, for if

$$
iw_k = w_mg_{r_1}
$$

and

$$
iw_{k_1} = w_mg_{r_1},
$$

where the w's are coset multipliers in (1) and the g's are in G, then

$$w_k = i^{-1}w_m g_{r_j},$$

and

$$w_{k_1} = i^{-1}w_m g_{r_j} = g_{r_n} i^{-1} w_m g_{r_i} = g_{r_n} w_k,$$

where g_{r_n} is an element of G such that $g_{r_n} = g_{r_j}$. This contradicts theorem B unless w_k is w_{k_i}. Therefore, by (2) and (4)

$$S((iw_1)^a, (iw_2)^a, \cdots, (iw_k)^a) = S = i^{ad}S,$$

and

$$(i^{ad} - e)S = 0.$$

Since $i^{ad} - e$ is a unit, it is not a divisor of 0. Hence, $S = 0$.

We can easily illustrate our theorem. Consider the ring R of residue classes modulo 21. It has as a divisor group G of order three of units, the group whose least positive residues are 4, 16, and 1, and a cyclic subgroup of order two of units, generated by the residue class whose least positive residue is 20. And $20^3 - 1 \equiv 19 \pmod{21}$, where 19 is the least positive residue of a unit. A complete set M of coset multipliers of R with respect to G has as least positive residues the integers 1, 2, 3, 5, 7, 9, 10, 14, and 21. In the terminology of our theorem, $a = 3$, $n = 2$. Let $d = 1$ and $S \equiv \sum w_i^3 \pmod{21}$, where w varies over M. Then it is easy to verify that $S \equiv 0 \pmod{21}$.

Again let R be a Galois field with unity 1 and order p^n, and let a divide $p^n - 1$ and d be an integer such that $p^n - 1$ does not divide ad. Then R contains a multiplicative divisor group G of order a and a multiplicative cyclic subgroup H of order $p^n - 1$. Let the generator of H be i; then $i^{ad} - 1$ is a unit since $p^n - 1$ does not divide ad. Hence, an IHSF of degree d of the a-th powers of a complete set of coset multipliers of R with respect to G is zero in R unless $p^n - 1$ divides ad. This is theorem IV of Vandiver's paper [9].

COROLLARY. *Let r vary over the set A: $1, 2, \cdots, p-1$ where p is an odd prime, and let d not be a multiple of $p-1$, k be prime to p, and t be an integer; then each IHSF of degree d of the p^{n-1}-st powers of the integers of the set B: $kr + tp$ is congruent to zero modulo p^n.*

Proof. We shall show that this corollary is a special case of the principal theorem. Let G be the cyclic [2] multiplicative subgroup of units modulo p^n of order $\phi(p^n) = p^{n-1}(p-1)$; and let g be a primitive root modulo p^n. Then there exists a subgroup H of order p^{n-1}; also there exists an element $g^{p^{n-1}}$ of order $p-1$; furthermore $g^{dp^{n-1}} - 1$ is a unit modulo p^n, otherwise $g^{dp^{n-1}} \equiv 1 \pmod{p}$ which contradicts the fact that d is not a multiple of $p-1$, for g is also a primitive root modulo p. To show that the elements of the set B are a complete set of coset multipliers of G with respect to H, we first notice that the elements of H are:

$$g^{vp},\ v = 1, 2, \cdots, p^{n-1}.$$

Certainly each element of the set B is an element of G. If

$$(kr_j + tp)g^{v(p-1)} \equiv kr_m + tp \pmod{p^n}$$

for r_j and r_m elements of A, and $1 \leqq v \leqq p^{n-1}$, then since $g^{v(p-1)} \equiv 1 \pmod{p}$ and $(k, p) = 1$, it follows that $r_j \equiv r_m \pmod{p}$, and $r_j = r_m$; hence by theorem B and Lagrange's theorem, the set B is a complete set of the coset multipliers of G with respect to H. Let R be the ring modulo p^n. One complete set C of the coset multipliers of R with respect of H is therefore the set B together with a set D of non-units. Then by our main theorem each IHSF of degree d of the p^{n-1}-st powers of the elements of C is 0 in the ring. Denote by $s_i(T)$ the sum of the products taken i at a time of the elements of a given set T. Let S be an IHSF of degree d of the p^{n-1}-st powers of the elements of the set B. Then S is expressible [8] as $S \equiv f(s_1(B'), s_2(B'), \cdots, s_{p-1}(B')) \pmod{p^n}$ where B' is the set of p^{n-1}-st powers of the elements of the set B. But $f \equiv g(s_1(C'), s_2(C'), \cdots, s_{p-1}(C')) \pmod{p^n}$ where C' is the set of p^{n-1}-st powers of the elements of the set C, since the p^{n-1}-st power of a non-unit of C is zero in R. Since S may be expressed as the sum of the non-zero terms of the IHSF g which is zero in R, it follows that $S \equiv 0 \pmod{p^n}$, and the corollary is proved.

If in our corollary $k = n = 1$ and $t = 0$, we obtain the theorem of Hensel, mentioned earlier.

If in our corollary $k = 1$, $n = 2$, $t = 0$, and $d = p^2 - p - 1$, we consider the function $S \equiv \sum r^{p(p^2-p-1)} \pmod{p^2}$ and find that $S \equiv 0 \pmod{p^2}$. This is the theorem already mentioned which resembles that of Wolstenholme; for the latter theorem may be expressed as $\sum r^{p^2-p-1} \equiv 0 \pmod{p^2}$ for $p > 3$.

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STACKING COLORED CUBES

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1. Introduction. The six faces of a cube can be painted with six different colors in thirty different ways. Thirty cubes so painted not only are the basis for a number of interesting games and puzzles, but also are excellent models for illustrating group theory, combinatorial counting, and cataloguing. This paper generalizes a puzzle of W. W. R. Ball [1] and extends the knowledge of these thirty cubes. It is hoped these results will help mathematicians make more use of this simple, attractive, yet complicated model. Possible applications to the theory of cubic crystal alloys also exist.

We set aside one cube, called the "key" cube. The three main theorems discussed are the following:

THEOREM I. *There are* 144,500 *different ways in which eight of the twenty-nine non-key cubes can be selected and stacked to form a* $2\times2\times2$ *cube such that* (1) *each face of the* $2\times2\times2$ *cube is a solid color, and* (2) *the colors have the same arrangement as those on the key cube.*

THEOREM II. *If a set of eight non-key cubes will stack properly, it will do so in* 2, 4, 8 *or* 16 *ways.*

THEOREM III. *There are* 67,260 *different sets of eight non-key cubes that will stack properly.*

These numerical results are valuable for some of the games and puzzles. However, actually proving these results, recognizing differences and avoiding duplications in the counting process, is even more interesting.

To fix our ideas, we call the colors A, B, C, D, E, F. Let us suppose the key cube to be the cube with A on top, F on the bottom, and $BCDE$ on the sides in counterclockwise order looking down. We call it $ABCDEF$.

The secret of the proof is to concentrate on the key corners rather than on the cubes. A key corner is one like one of the eight corners of the key cube. ABC is a key corner, with the colors listed in counterclockwise order as seen by an external observer looking directly at the corner.

Corner ABC lies on five different cubes. For, after three faces are painted A, B, and C, the remaining three can be painted D, E, and F in six different ways, forming six different cubes. One of these is the key cube, leaving five among the 29.

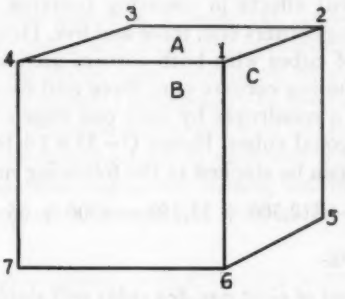
No cube can have three key corners except the key cube. For, if a cube has three key corners, two of them must be diagonal corners of one of the faces of the key cube. This would require that five of the faces be colored like the key cube, and hence all six.

A careful look at the five non-key cubes with corner ABC shows that each has exactly one other key corner as well. $ABCFDE$ and $ABCEFD$ have the key corner DFE. We note that ABC and DFE are end points of a solid diagonal of

the key cube. The other three cubes, $ABCDFE$, $ABCFED$, $ABCEDF$ have the three key corners ACD, AEB, BFC respectively. These three corners, with ABC, are the end points of the three edges of the key cube which meet at ABC.

Symmetry shows similar results for the other key corners. Hence the 29 cubes divide into three groups: (1) eight cubes with two key corners which are ends of the solid diagonals of the key cube, two for each of the four solid diagonals, (2) twelve cubes with two key corners which are ends of the twelve edges of the key cube, one for each edge, and (3) nine cubes with no key corners at all. The cubes of groups (1) and (2) are called "diagonal" and "edge" cubes respectively. It is easily seen that the nine cubes of group (3) are the mirror images of the key cube and the eight diagonal cubes of group (1).

2. Counting the number of ways eight cubes can be stacked. Clearly it is possible to find eight cubes which will stack. All one needs to do is to pick out eight key corners on eight different cubes. In particular, the eight diagonal cubes of group (1) will stack. However, not every set of eight cubes will stack. For example, if a cube of group (3) is included, the set will not stack, since this cube will fit nowhere.



To assist in the counting, the key corners are numbered according to the diagram in the figure. With this system every edge is bounded by one odd and one even numbered corner, and each solid diagonal is bounded by corners whose sum is nine. A cube with key corners is now named by the numbers of its two corners. The twelve edge cubes of group (2) are named 12, 14, 16, 32, 34, 38, 52, 56, 58, 74, 76, 78. The eight diagonal cubes of group (1) are named 18, 18′, 36, 36′, 54, 54′, 72, 72′, where prime merely distinguishes one of the two cubes which have the given diagonal key corners.

Since each corner appears on five different cubes, every freshman knows there are 5^8 or 390,625 different sets of eight different key corners. If each of these sets appeared on eight different cubes, this would be the number of ways of stacking eight cubes. Unfortunately, among these sets of corners will be those where both key corners of one, two, three, or even four cubes have been used. These sets of corners lie on less than eight cubes, which, therefore, do not stack. We subtract the number of such sets.

Let S, P, T, Q be the number of times both corners of a single, a pair, a triple, a quadruple of cubes are used. For example, consider the set of eight corners where corners one and two are taken from cube 12, corners three and four are taken from cube 34, and the remaining four corners from four different cubes. This set contributes two to the value of S and one to the value of P.

The number of different sets of eight corners on less than eight different cubes is $S-P+T-Q$. The total number of ways eight cubes can be stacked is $390{,}625-S+P-T+Q$.

Clearly both corners of the cube 12 are used in 5^6 sets, 5^6 being the number of ways the remaining six corners could be chosen after corners one and two are assigned to cube 12. Symmetry shows that the same is true for each cube. Hence $S=20\times 5^6=312{,}500$.

Suppose both corners of a pair of cubes are used which include corners one and three. There are 19 such pairs, as is easiest seen by cataloguing. Each such pair is used in 5^4 sets of 8 corners. Since there are 6 pairs of odd corners like one and three, $P=19\times 5^4\times 6=71{,}250$.

In counting T and Q some caution is required because edge cubes and diagonal cubes have different effects in counting patterns. However, there are 49 triples of cubes including corners one, three and five. Hence $T=49\times 5^2\times 4=4900$.

Every quadruple of cubes with both corners used will contain a triple with both corners used including corners one, three and five. Thirty-three of the 49 such triples are made a quadruple by only one edge cube. Sixteen are made a quadruple by two diagonal cubes. Hence $Q=33\times 1+16\times 2=65$.

Hence eight cubes can be stacked in the following number of ways:

$$390{,}625 - 312{,}500 + 71{,}250 - 4900 + 65 = 144{,}500.$$

3. Multiple stacking.

THEOREM II. *If a set of eight non-key cubes will stack properly, it will do so in* 2, 4, 8 *or* 16 *ways.*

Proof. Every cube which is used has two key corners. Draw the eight line segments connecting these key corners. These line segments will form a space polygon which will contain one, two, three, or four closed polygons. For example, the cubes 12, 23, 34, 41 would yield a closed polygon. So would the cubes 18, 18′, although this polygon is somewhat degenerate, having but two sides, one on top of the other.

The set of cubes forming a closed polygon can be placed in two positions in stacking corresponding to the two senses of the polygon. This is the only way cubes may be stacked in more than one manner. Hence the cubes can be stacked in 2, 4, 8, 16 different ways as their space polygon has one, two, three or four closed polygons.

The eight diagonal cubes are the only set which will stack in 16 different ways. This set is also the only set which can be stacked with all abutting faces the same color. They are thus the solution to Ball's puzzle mentioned above.

4. Sets of cubes which will stack. The variety of ways sets of eight cubes will stack suggests the question "How many different sets of 8 cubes from the 29 will stack?" The answer is in the following theorem.

THEOREM III. *There are* 67,260 *different sets of eight non-key cubes that will stack properly.*

The proof of theorem III is somewhat abbreviated. Eight cubes from the set of twenty will stack unless (I) one or more key corners is missing, or (II) some, say x, of the key corners appear on fewer than x cubes, or (III) both. If I, II, and III also represent the numbers of sets of eight cubes respectively described, then the number of sets which stack is $C(20, 8)-\mathrm{I}-\mathrm{II}+\mathrm{III}$ where $C(20, 8)$ is the standard symbol for the combinations of 20 objects 8 at a time.

Sets of eight cubes exist with one, two, or even three key corners missing. In counting these it is necessary to be aware of the different arrangements of the missing corners. For example, two missing key corners might be ends of an edge, a face diagonal or a solid diagonal of the key cube. It turns out that single corners are missing 51,480 times, pairs of corners are missing 4500 times, and triples of corners are missing 24 times. Hence there are $51{,}480-4500+24=47{,}004$ sets of eight cubes with one or more key corners missing.

There are sets of eight cubes where two key corners appear on only one cube, three corners on only two cubes, four corners on only three cubes, and four corners on only two cubes. Again it is necessary to keep account of whether the doubled cubes are edge or diagonal cubes, and how the corresponding edges and diagonals are related. The total number of such sets is 12,378.

There are also 672 sets with one corner missing and two corners on one cube, or three corners on two cubes. Hence the total number of cubes which will stack is

$$C(20,8) - 47{,}004 - 12{,}378 + 672 = 67{,}260.$$

Two interesting corollaries are the following. The probability that eight cubes will stack when chosen at random from the twenty edge and diagonal cubes is $67{,}260/C(20, 8)=.54$. The probability that eight cubes will stack when chosen from 29 (all except the key cube) is $67{,}260/C(29, 8)=.016$.

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A NATIONAL WEAKNESS

Editorial note. This article is reprinted from the *Quarterly Report of the Carnegie Corporation of New York*, vol. 4, 1956.

Concern over the mathematical incompetence of the average—and even above-average—American has become almost a national preoccupation. Science and industry cry in vain for more and better mathematicians. Ordinary businesses ask only that their employees be able to do simple arithmetic. Neither the extravagant nor the modest demands of society for mathematicians—or arithmeticians even—are being met. And public concern grows.

A recent study shows that there are more than adequate explanations for this national weakness, and that many of them lie in the caliber of mathematical instruction. Under a grant from Carnegie Corporation, the Educational Testing Service (ETS) recently conducted a survey to identify some of the reasons why so many elementary and secondary school children either fail or drop mathematics or, at best, just manage to limp through. Here are some of the basic facts which inspired the ETS to undertake its study:

A recent national survey showed that since 1910 there has been a consistent decline in the proportion of high school students who take college preparatory mathematics. It is true that the character of the high school population has changed with the vast increase in numbers. But this does not explain away the fact that many students starting mathematics courses in high school do not see them through. Some drop elementary algebra before they finish the first year; more quit in the middle of geometry; still more leave intermediate algebra unfinished. These facts would suggest a certain indifference toward the subject on the part of many youngsters; actually in many cases it is positive detestation rather than mere indifference. Almost all available data show that students have a poorer attitude toward math than toward any other school subject; in one survey 40 per cent of the pupils honored math by electing it the subject they dislike most.

Another basic fact which impressed the ETS is that mathematical incompetence is widespread even among students of superior intellectual ability. In one high school, of 526 students with I.Q.'s above 114, 135 either dropped math, averaged C or lower, or were retarded one grade or more. The mean I.Q. of these 135 students was 123.

Faced with these bewildering and discouraging figures, the ETS committee assembled data and expert opinions on the two important factors which many people believe lie at the heart of the difficulty: teaching and curriculum. They surveyed the relevant literature, sent questionnaires to experts in a variety of fields, observed classroom procedures, and interviewed students. From all these sources, they learned some astonishing facts about math teaching in the United States today.

For instance, although all states require *education* courses for secondary

mathematics teachers, a third of the states require no *mathematics* for certification of math teachers. At the elementary level the situation is even worse. In the majority of instances a prospective elementary school teacher can enter a teachers college without any credits in secondary school math. In most states, a teacher can be certified to teach elementary school math without any work in math at the college level.

Under such circumstances, it is no surprise that one professor states: "Elementary teachers, for the most part, are ignorant of the mathematical basis of arithmetic. . . . " As for the secondary school teachers, one math professor interested in teacher education says "They are not as good as our run-of-mine juniors."

An observer who visited 60 representative math classrooms in different regions of the country came to the conclusion that genuine and efficient mathematical learning was going on in only eight of them. He got the inescapable impression from the 36 elementary classrooms he observed that in most cases the instruction is carried on in such a routine, haphazard way that learning of any kind is probably largely accidental. And his observation of 24 high school classrooms led him to think that teaching on that level is no better, and perhaps even worse, than it is on the elementary level.

As for the curriculum, little change has occurred in it despite extensive changes in the nature and knowledge of mathematics and despite the fact that since 1894 many authorities have been recommending changes. The result is that the curriculum is out of touch with the real needs for math in the world today, and also out of touch with the interests of the pupils. The only major development has been the growth in a ninth-grade "general mathematics" course, a subject that the students tend to regard with disdain but often take, as the lesser of two evils, rather than algebra.

Although the total picture of mathematical education that emerges from the ETS survey is hardly encouraging, there are some moves toward reform, particularly in the curriculum. Further reforms may be instituted as the educators and public who must meet the problems are provided with information, such as that contained in the ETS survey, about the actual situation today. To give wide distribution to these facts, the ETS will soon publish a pamphlet summarizing the findings of the survey. The hope is that circulation of these findings will stimulate schools to re-examine their programs in mathematics to see what measures might be taken to improve them.

COMMENTARY: MATHEMATICS IN CRISIS

There is grave cause for alarm in the failure of our schools to provide more youngsters with a reasonable command of mathematics. It is a problem which merits the attention of all men and women accustomed to concerning themselves about the future of America.

In recent years, the astonishing advances in science and technology and the prompt application of these advances to our industrial and national defense

needs have created an unprecedented demand for intelligent and highly trained young men and women. It is now clear that unless heroic measures are taken the demand simply cannot be met.

What we are facing is not a shortage of talent, but of *trained talent*. Intelligent youngsters are not in short supply, and if we had made good use of the resources available to us we would be in no difficulty. This is not an easy fact for us to grasp. Throughout history, human societies have managed to be extremely wasteful of individual talent. The discovery and nurture of talent has been largely accidental. The needs of society have not in the past put a premium either upon gifts or upon training. Even a generation ago the world did not reckon highly intelligent, highly trained men among the most marketable commodities. Now for the first time in history a radically different condition exists. The nation needs its talented and highly trained men and women, and it needs them badly.

The national need for men with scientific and mathematical competence exists at all levels. It is not just that we need more creative scientists at the Nobel Prize level. Behind the great creative minds in science moves an army of able and superbly trained men who test and confirm (or reject) new discoveries, conduct the more mundane testing and experimenting of the scientific world, and make the numberless lesser contributions on which the structure of science rests. And beside them works an immense corps of well-trained and skillful laboratory technicians and assistants.

In the industrial world a similar situation exists. In the spotlight are the great designers, inventors, and innovators, but back of them stands an immensely able corps of technically brilliant men to put their discoveries into operation. And back of these men stretch rank after rank of highly trained technicians and mechanics. Only in recent years have we learned that much of the strength and vitality of American technology is in this "training in depth." Several of the leading countries of the world can match us for brilliant men at the top. But we outstrip most of them in competent, excellently trained second, third, fourth, and fifth level workers.

When it became apparent two or three years ago that we were going to experience a severe shortage of scientists and engineers, and that this shortage constituted cause for the gravest national concern, some first-rate minds were put to work to diagnose the problem. Why the shortage? How can it be corrected?

There are many reasons for the shortage, and these need not concern us here. When the experts turned to the question of how to correct it, they very soon were brought up short before the great stumbling block of mathematics. You cannot make scientists out of youngsters who have not been adequately trained in the early years of mathematics. And the youngsters who had received such adequate training proved to be all too rare.

That was the situation which led to the study described in the accompanying article, "A National Weakness." It would be foolish in the extreme to waste our

time and energy in fixing the blame for this disturbing state of affairs. Whatever critics may say, our schools as a whole are *not* deteriorating. They are doing a good job under extraordinarily difficult circumstances. Any attempt to find a scapegoat for the present situation will simply waste everyone's emotional energies, and divert us from constructive action.

But action is necessary. We must act to draw a vastly greater number of first-rate young men and women into the teaching of mathematics. We must act to strengthen and upgrade many of those who are already in it. And we must improve the training of those who teach.

A few months ago Carnegie Corporation made a grant of $300,000 to the American Association for the Advancement of Science to support a program for the improvement of teaching in science and mathematics. The ETS study reported here is another effort to draw national attention to a truly critical problem.

JOHN W. GARDNER

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

THE ASYMPTOTIC DENSITY OF SOME k-DIMENSIONAL SETS

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Let A be a k-dimensional set whose elements are k-tuples of positive integers, and let $A(n)$ be the number of k-tuples, $(a_1, \cdots, a_k)$, in A such that $a_i \leqq n$ for $i=1, 2, \cdots, k$. Then $\delta(A)=\lim_{n\to\infty}\{A(n)/n^k\}$ is defined to be the *asymptotic density of the set* A. It can also be said that $\delta(A)$ represents the *probability* that any k-tuple of positive integers, chosen at random, will be in the set A.

Suppose k positive integers ($k\geqq 2$) are chosen at random. What is the probability that they have greatest common divisor t, where $t\geqq 1$? Or the probability that t be square free? These questions will be answered in this note. For $k=2$ and $t=1$, Mertens [3] showed that the probability is $1/\zeta(2)$ where $\zeta(s)$ is the well known Riemann-Zeta function. For $k=1$, Gebenbauer [1] has shown that the probability that an integer be square free is also $1/\zeta(2)$.

Definition. Let $\phi_k(n)$ *be the number of integers,* a, *such that* $0<a\leqq n$ *and* $(a, n)=k$, *where* (a, n) *denotes the greatest common divisor of* a *and* n. It is easy to show that $\phi_k(kn)=\phi(n)$, where $\phi(n)$ is the well known Euler ϕ-function. We are now ready to state

THEOREM 1. *Let* t *be a positive integer. Then the probability that two positive*

integers, selected at random, have greatest common divisor t is $1/t^2\zeta(2)$.

Proof. There exist t^2n^2 pairs of numbers (a, b) such that $0<a\leqq tn$ and $0<b\leqq tn$. Let N_n be the number of these pairs for which $(a, b)=t$. Then, by counting, it is seen that

$$N_n = \phi_t(t) + 2\phi_t(2t) + \cdots + 2\phi_t(nt).$$

And so, by the remark following the above definition,

$$N_n = 2[\phi(1) + \phi(2) + \cdots + \phi(n)] - 1.$$

But it is well known (see *e.g.*, Hardy and Wright [2]) that

$$\phi(1) + \phi(2) + \cdots + \phi(n) = \frac{3}{\pi^2} n^2 + O(n \log n).$$

Thus

$$\lim_{n\to\infty} \frac{N_n}{t^2n^2} = \frac{6}{t^2\pi^2} = \frac{1}{t^2\zeta(2)}$$

which is the desired probability.

THEOREM 2. *Let a_1 and a_2 be two positive integers selected at random. The probability that their greatest common divisor, (a_1, a_2), be square free is $1/\zeta(4)$.*

Proof. In the ordinary Cartesian plane consider all pairs of integers, (x, y), in the square $0<x\leqq n$ and $0<y\leqq n$. Call this square Q_n, and note that Q_n contains n^2 pairs of positive integers. By Theorem 1 and its proof, it is seen that Q_n contains

$$\frac{1}{t^2\zeta(2)} n^2 + O\left(\frac{n}{t} \log \frac{n}{t}\right)$$

pairs which have greatest common divisor t. Let N be the number of pairs in Q_n for which t is square free. Then

$$N = \sum_{t=1}^{n} \left\{ \frac{|\mu(t)|}{t^2\zeta(2)} n^2 + O\left(\frac{n}{t} \log \frac{n}{t}\right)\right\}$$

where $\mu(t)$ is the well known Möbius function. Now

$$\lim_{n\to\infty} \frac{N}{n^2} = \frac{1}{\zeta(2)} \sum_{t=1}^{\infty} \frac{|\mu(t)|}{t^2} + \lim_{n\to\infty} \frac{\sum_{t=1}^{n} O\left(\frac{n}{t} \log \frac{n}{t}\right)}{n^2}$$

$$= \frac{1}{\zeta(2)} \cdot \frac{\zeta(2)}{\zeta(4)} = \frac{1}{\zeta(4)}.$$

Three theorems are now stated which are the generalizations of Mertens' result, Theorem 1, and Theorem 2 to k-dimensions.

THEOREM 3. *If $k \geqq 2$, then the probability that k integers, selected at random, are relatively prime is $1/\zeta(k)$.*

THEOREM 4. *Let t be a positive integer. If $k \geqq 2$, then the probability that k integers, selected at random, have greatest common divisor t is $1/t^k\zeta(k)$.*

THEOREM 5. *Let $a_1, \cdots, a_k$ be $k(k \geqq 1)$ positive integers selected at random. The probability that the greatest common divisor, $(a_1, \cdots, a_k)$, of these integers be square free is $1/\zeta(2k)$.*

Note that when $k=1$, Theorem 5 becomes Gebenbauer's result, and that all three theorems have been established for the case $k=2$. By a straightforward induction on k they may be established for all integers k.

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A BASIC SET OF POLYNOMIAL SOLUTIONS FOR THE EULER-POISSON-DARBOUX AND BELTRAMI EQUATIONS

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The equations

$$E_{ij}\colon \sum_{n=1}^{m} u_{x_n x_n} + (-1)^i[u_{tt} + jkt^{-1}u_t] = 0 \qquad i = 0, 1; j = 0, 1$$

are called: E_{10}, the wave equation, E_{00}, the Laplace equation, E_{11}, the Euler-Poisson-Darboux equation and E_{01}, the Beltrami equation. In a recent article [1] the authors presented a basic set of polynomial solutions for $E_{10}[E_{00}]$; in this note we modify certain of these polynomials to obtain the basic set of polynomial solutions for $E_{11}[E_{01}]$ associated with the Cauchy problem for $E_{11}[E_{01}]$ where $k>0$.

The following lemma is used to carry out the modification.

LEMMA. *Let $H_{10}[H_{00}]$ be a polynomial solution of $E_{10}[E_{00}]$ which is even in t; then the polynomial $\{H_{10}\}[\{H_{00}\}]$ obtained from $H_{10}[H_{00}]$ by replacing t^{2n} by*

$$t^{\{2n\}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(1+k)(3+k)\cdots(2n-1+k)} t^{2n}$$

is a solution of $E_{11}[E_{01}]$.

Proof. In operator notation we may write $E_{ij} \sim [\nabla^2 + D_{ij}]u = 0$. We note that

$$D_{10}[t^{2n}] = -2n(2n-1)t^{2n-2}, \quad \text{and} \tag{1}$$

$$\begin{aligned} D_{11}[t^{\{2n\}}] &= -\left[\frac{2n(2n-1)\cdot 1\cdot 3\cdot 5\cdots(2n-1)}{(1+k)(3+k)\cdots(2n-1+k)}\right. \\ &\quad \left. + \frac{k\cdot 2n\ 1\cdot 3\cdot 5\cdots(2n-1)}{(1+k)(3+k)\cdots(2n-1+k)}\right] t^{2n-2} \qquad (2) \\ &= -2n(2n-1)\frac{1\cdot 3\cdot 5\cdots(2n-3)}{(1+k)(3+k)\cdots(2n-3+k)}t^{2n-2} \\ &= -2n(2n-1)t^{\{2n-2\}}. \end{aligned}$$

Any H_{10} may be written as $\sum_{s=0}^{p} P_s[x]t^{2s}$ where each P_s is a polynomial in $x_1, \cdots, x_m$. Since H_{10} is a solution of E_{10} we have

$$[\nabla^2 + D_{10}]H_{10} = \sum_{s=0}^{p} Q_s[x]t^{2s} \text{ and} \tag{3}$$

$$Q_s[x] = 0, \qquad s = 0, 1, \cdots, p. \tag{4}$$

Writing

$$\{H_{10}\} = \sum_{s=0}^{p} P_s[x]t^{\{2s\}}$$

and making use of (1), (2), (3) and (4), we see immediately

$$[\nabla^2 + D_{11}]\{H_{10}\} = \sum_{s=0}^{p} Q_s[x]t^{\{2s\}} = 0, \tag{5}$$

which completes the proof of the unbracketed form of the lemma. The bracketed form follows in similar fashion.

The authors' basic set [1] for E_{10} may be given as follows. For any set of non-negative integers (b_j) such that $b_0 \leqq 1$ and $\sum_{j=0}^{m} b_j = n$, let $H^{n}{}_{b_0, b_1, \cdots, b_m}(x_1, x_2, \cdots, x_m, t)$ denote

$$\sum \frac{\left[\frac{a_0}{2}\right]!}{\prod_{j=0}^{m} a_j! \prod_{j=1}^{m}\left[\frac{b_j - a_j}{2}\right]!} t^{a_0} \prod_{j=1}^{m} x_j^{a_j} \tag{6}$$

where the summation is extended over all non-negative (a_j) such that

$$\begin{aligned} &(1)\ a_j \equiv b_j \bmod 2, && j = 0, 1, 2, \cdots, m \\ &(2)\ \sum_{j=0}^{m} a_j = n \qquad (7) \\ &(3)\ a_j \leqq b_j, && j = 1, 2, \cdots, m. \end{aligned}$$

The polynomials (6) are even in t if $b_0=0$ and odd in t if $b_0=1$. We can readily see that no polynomial solution of $E_{11}[E_{01}]$ contains an odd power of t if $k>0$. For suppose the coefficient $P_n[X]$ of t^{2n+1} is the first nonvanishing coefficient of an odd power of t occurring in $P(x, t)$, a polynomial solution of E_{11}. The coefficient of t^{2n-1} in $(\nabla^2+D_{11})P(x, t)$ would then be $-P_n[x]\cdot(2n+1)(2n+k)$, a nonvanishing function, since k being positive cannot equal $-2n$, $n\geqq 0$.

We form our basic set for E_{11} from (6) by restricting b_0 to the value 0 and replacing t^{a_0} by $t^{[a_0]}$. That the resulting polynomials

$$(8)\quad H^n_{0b_1,\cdots,b_m}(x_1, x_2, \cdots, x_m, \{t\}) = \sum \frac{\left(\frac{a_0}{2}\right)!}{\prod_{j=0}^{m} a_j! \prod_{j=1}^{m}\left(\frac{b_j-a_j}{2}\right)!}\, t^{[a_0]} \prod_{j=1}^{m} x_j^{a_j}$$

are solutions of E_{11} follows from our lemma.

Since a particular monomial $\prod_{j=1}^{m} x_j^{b_j}$ occurs only in the polynomial (8) determined by its exponent set 0, $b_1, \cdots, b_m$ we readily conclude that the polynomials (8) are independent. We call this monomial the generator for the associated polynomial. To show that this set of independent homogeneous polynomials is a basic set of solutions we need only to show that it contains the correct number of members. The polynomials (8) of degree n are

$$\binom{n+m-1}{m-1}$$

in number since they are generated in turn by the individual monomials appearing in $[\sum_{j=1}^{m} x_j]^n$.

Let us write a general homogeneous polynomial of degree n in the $m+1$ variables $t, x_1, \cdots, x_m$ which contains no first powers of t, as $P^n(t, x_1, \cdots, x_m) = P_1^n(x_1, \cdots, x_m)+t^2P_2^{n-2}(t, x_1, \cdots, x_m)$; thus P^n has

$$\binom{n+m-1}{m-1}+\binom{n+m-2}{m}$$

coefficients. Equating $(\nabla^2+D_{11})P^n$ to zero gives exactly

$$\binom{n+m-2}{m}$$

independent conditions on these coefficients, leaving

$$\binom{n+m-1}{m-1}$$

of them arbitrary. Hence the number of independent homogeneous polynomial solutions of degree n for E_{11} is

$$\binom{n+m-1}{m-1}$$

and the polynomials (8) are a basic set for E_{11}, $k>0$.

Since the polynomials (6′) obtained from (6) by replacing t^{a_0} by $(-1)^{[a_0/2]}t^{a_0}$ form a basic set [1] for E_{00}, we readily observe that the polynomials (8′) obtained from (8) by replacing $t^{[a_0]}$ by $(-1)^{(a_0/2)}t^{[a_0]}$ form a basic set for E_{01}.

We point out the association of the basic set (8) with the solution of the analytic Cauchy problem for E_{11}:

$$u_t(x_1, \cdots, x_m, 0) = 0$$

$$u(x_1, \cdots, x_m, 0) = f(x_1, \cdots, x_m) = \sum_{n=0}^{\infty}\Big[\sum A_{b_1,b_2,\cdots,b_m}\prod_{j=1}^{m} x_j^{b_j}\Big]$$

where the bracketed summation is over all non-negative b_j, $\sum_{j=1}^{m} b_j = n$. A formal solution is seen to be

$$u(x_1, \cdots, x_m, t) = \sum_{n=0}^{\infty}\Big[\sum A_{b_1,b_2,\cdots,b_m}\prod_{j=1}^{m}(b_j)!H^{n}_{0,b_1,\cdots,b_m}\Big]. \tag{9}$$

As an illustration we write, using (9), the solution, known to be unique [2], for the Cauchy problem for the E. P. D. equation

$$u_{xx} + u_{yy} - (u_{tt} + kt^{-1}u_t) = 0, \qquad k > 0;$$

$$u(x, y, t) = x^2y^2;\ u_t(x, y, 0) = 0.$$

The solution is

$$u(x, y, t,) = x^2y^2 + \frac{(3xy^2 + x^3)t^2}{1+k} + \frac{3xt^4}{(1+k)(3+k)}. \tag{10}$$

The solution (10) is valid for $k<0$, $k \neq -1, -3$ but is not unique.

Note added in proof. A theory which gives these and certain more general results in a simpler manner has been developed in the authors' paper, "The Cauchy problem for linear partial differential equations with restricted boundary conditions", to appear shortly in the *Canadian Journal of Mathematics*.

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ON MATRIC SOLUTIONS OF A CYCLIC EQUATION

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A. R. Richardson* has exhibited a set of matric conjugates M_i of the cyclic equation $x^3+x^2-2x-1=0$, such that M_{i+1} is obtained from M_i by cyclically permuting rows and columns. These conjugates arose incidentally and no comment was made on this particular property of the matrices. It is possible to show that the occurrence of such a set was not a coincidence, but that every cyclic equation possesses a set of conjugate matric solutions with this property.

THEOREM. *If $f(x)=0$ is a cyclic equation over the rational field, there exists a conjugate set of rational matric solutions B_i such that B_{i+1} is obtained from B_i by cyclicly permuting rows and columns.*

Let the complex roots of $f(x)=0$ be $x_1, x_2, \cdots, x_n$, where $x_{i+1}=\theta(x_i)$. Then if A_1 is the companion matrix of $f(x)$, the matrices A_1, $A_2=\theta(A_1), \cdots, A_n=\theta(A_{n-1})$ form a conjugate set of rational matric solutions. If $H=VUV^{-1}$, where V is the Vandermonde matrix (x_j^{i-1}) and U is obtained from the identity matrix of order n by cyclically permuting columns, then H is rational and of determinant $(-1)^{n-1}$. Albert† has shown that $H^{-1}A_1H=A_2$ whence, in general, we have

$$H^{-1}A_iH = A_{i+1}, \qquad i = 1, 2, \cdots, n-1. \tag{1}$$

While the relation $H=VUV^{-1}$ does not establish the similarity of H and U in the rational field, there does exist a rational matrix T such that‡

$$T^{-1}HT = U. \tag{2}$$

Now let $B_i=T^{-1}A_iT$. It follows immediately that the B_i constitute a conjugate set of rational matrices. Moreover, from (1) and (2), we have

$$\begin{aligned} B_{i+1} &= T^{-1}A_{i+1}T \\ &= T^{-1}H^{-1}A_iHT \\ &= T^{-1}H^{-1}TB_iT^{-1}HT \\ &= U^{-1}B_iU. \end{aligned}$$

But the transformation $U^{-1}B_iU$ has the effect of cyclically permuting the rows and columns of B_i, and the theorem is proved.

The set B_i having this property is by no means unique since it is not necessary that the transforming matrix H be as defined above. For example, when the equation $f(x)=0$ is cubic, the general matrix H for which (1) holds has its elements expressible in terms of three parameters and, in general, it is possible to choose these in a variety of ways so that the determinant of H is unity. For any

* Quarterly Journal of Mathematics, Oxford Series, vol. 7, 1936, pp. 256–270.

† Modern Higher Algebra, University of Chicago Press, 1937, p. 241.

‡ *Ibid.*, p. 77, corollary.

such H, the general matrix T satisfying (2) is also a three parameter matrix so that T may be constructed in an infinite number of ways. Thus, for the cyclic cubic equation $x^3+x^2-2x-1=0$, where

$$A_1=\begin{bmatrix}-1 & 2 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\quad A_2=A_1^2-2=\begin{bmatrix}1 & -1 & -1\\ -1 & 0 & 1\\ 1 & 0 & -2\end{bmatrix},$$

$$A_3=A_2^2-2=\begin{bmatrix}-1 & -1 & 0\\ 0 & -1 & -1\\ -1 & -1 & 1\end{bmatrix},$$

on setting

$$H=\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}$$

and solving the system of equations obtained by equating corresponding elements of $A_1H=HA_2$, for a, b, c, d, e, and g in terms of h, f, and i, we obtain

$$H=\begin{bmatrix}h-f-3i & f+i & -h+3i\\ -f-i & -h-f+2i & f\\ h-f-2i & h & i\end{bmatrix}.$$

The determinant of H is $7(i^3+i^2f-2i^2h+h^2i-hif)-f^3-h^3+2h^2f+hf^2$, and it is clear that this has the value unity for various choices of the parameters f, h, and i.

With either

$$H=\begin{bmatrix}1 & -1 & 0\\ 1 & -1 & -1\\ 1 & 0 & 0\end{bmatrix}\quad\text{and}\quad T=\begin{bmatrix}2 & 4 & 1\\ 1 & 2 & -3\\ 4 & 1 & 2\end{bmatrix}$$

or

$$H=\begin{bmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\quad\text{and}\quad T=\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix},$$

we obtain a set of symmetric matrices

$$B_1=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & -1\\ 0 & -1 & -1\end{bmatrix},\quad B_2=\begin{bmatrix}-1 & 0 & -1\\ 0 & 0 & 1\\ -1 & 1 & 0\end{bmatrix},\quad B_3=\begin{bmatrix}0 & -1 & 1\\ -1 & -1 & 0\\ 1 & 0 & 0\end{bmatrix},$$

while if

$$H = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix},$$

we are led to a set of non-symmetric conjugates

$$B_1 = \begin{bmatrix} -2 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & -2 & -2 \end{bmatrix}.$$

ON A FUNCTION OF RAMANUJAN

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Let

$$\phi(x) = \sum_1^\infty \frac{n^{n-2}}{(n-1)!} x^{n-1} e^{-nx}.$$

It has been conjectured by S. Ramanujan that

$$(-1)^k \frac{d^k \phi}{dx^k} \geqq 0, \qquad x \geqq 1, \qquad k = 1, 2, 3, \cdots,$$

and this conjecture has been proved for $k = 1, 2, 3, 4$ [2]. In this note we prove

THEOREM 1. *The function $x\phi(x)$ is not completely monotonic* for $x > 1$.*

Proof. Let

$$\beta(t) = 0, \qquad \text{for } 0 \leqq t < 1$$

$$= \sum_{1 \leqq n \leqq t} \frac{n^{n-2}}{(n-1)!} e^{-n}, \qquad \text{for } t > 1.$$

Since the series $\sum_1^\infty n^{n-2}e^{-n}/(n-1)!$ is convergent, $\beta(t)$ is bounded and non-decreasing for $t \geqq 0$. Further,

$$F(x) = \int_0^\infty e^{-xt} d\beta(t) = \sum_1^\infty \frac{n^{n-2}}{(n-1)!} e^{-n(x+1)}$$

is convergent for $0 \leqq x < \infty$ and so by Bernstein's theorem [3; p. 160] $F(x)$ is completely monotonic for $x \geqq 0$. Let $\psi(t) = t - \log(1+t)$; then $\psi(t)$ is real and continuous for $t \geqq 0$, $\psi(0) = 0$, and $\psi(t) > 0$ for $t > 0$, and $\psi(t) = \int_0^t h(t)dt$ where $h(t) = t/(1+t)$ is not completely monotonic in any interval $0 \leqq a \leqq t \leqq b$. Hence by a theorem due to Bochner and Schoenberg [1; pp. 43–44] it follows that

* For definition see [3; pp. 144–5].

$F(\psi(x))$ is not completely monotonic for $x>0$. But

$$F(\psi(x)) = \sum_1^\infty \frac{n^{n-2}}{(n-1)!} \exp\{-n(x - \log(1+x) + 1)\}$$
$$= (1+x)\phi(1+x).$$

Hence $(1+x)\phi(1+x)$ is not completely monotonic for $x>0$ and so $x\phi(x)$ is not completely monotonic for $x>1$.

Remark. If we take $\psi(t) = t + \log(1+t)$ we find that

$$\sum_1^\infty \frac{n^{n-2}}{(n-1)!} e^{-nx} x^{-n}$$

is completely monotonic for $x>1$.

THEOREM 2. *The function $e^x\phi(x)$ is not completely monotonic for $x>1$.*

Let $\beta(t)$ be a step function having jumps $(n+1)^{n-1}e^{-n}/n!$ at $t=n(n=1, 2, 3, \cdots)$ and consider

$$\int_0^\infty e^{-t(x-\log(1+x))}d\beta(t) = e^{(x-1)}\phi(x+1) - 1.$$

Hence by the previous argument, $e^{(x+1)}\phi(x+1)-1$ is not completely monotonic for $x>0$, and so $e^x\phi(x)-1$ is not completely monotonic for $x>1$. Now for $x \geqq 0$,

$$e^x\phi(x) \geqq 1 + xe^{-x}$$

and hence $e^x\phi(x)$ is not completely monotonic for $x>1$.

THEOREM 3. *The function $e^x\phi(x)$ is an absolutely monotonic function of $y=xe^{-x}$ for all x such that $0 \leqq xe^{-x} < e^{-1}$. Moreover,*

$$e^x\phi(x) = \sum_1^\infty \frac{n^{n-2}}{(n-1)!} y^{n-1},$$

and the last series is convergent for $0 \leqq y < e^{-1}$.

Since all the coefficients of the last series are positive, $e^x\phi(x)$ is an absolutely monotonic function of $y=xe^{-x}$ for $0 \leqq y < e^{-1}$, that is, for all x such that $0 \leqq x < 1$ and $1 < x \leqq \Delta$.

Remark. The function $x\phi(x)$ is an absolutely monotonic function of xe^{-x} for all x such that $0 \leqq x < 1$ and $1 < x \leqq \Delta$.

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A NOTE ON THE DIVIDING STREAM LINE IN HYDRODYNAMICS

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1. In the two-dimensional irrotational motion of an incompressible fluid past a fixed body, it is well known that the fluid meets the body at right angles to its boundary curve at a stagnation point [1]. By using the elements of vectorial analysis, the object of this note is to show that this result may be extended to three dimensional flows which are unsteady, rotational and compressible. We have the following result:

THEOREM. *In any fluid motion past a fixed solid body, the necessary and sufficient condition that the dividing stream line should meet the boundary surface of the solid at right angles is that any existing vorticity of the fluid at the point of impact should lie along the normal to the boundary surface.*

In this theorem no assumption is made as to the nature of the fluid, thus it is a general theorem on vector fields.

2. For convenience, denote the boundary surface by S and the point of impact by P. Let $\mathbf{h}$ be the unit outward-drawn vector normal to S at P. Let $\mathbf{q}$ be the velocity vector of the fluid, and form the dyadic $\nabla\mathbf{q}$ at P. We shall suppose that this dyadic does not vanish. (This will most certainly be the case if the fluid motion is rotational at P).

Since $\mathbf{q}=0$ at P, $\delta\mathbf{q}=\delta\mathbf{r}\cdot\nabla\mathbf{q}$ gives the velocity distribution in the neighbourhood of P, where $\delta\mathbf{r}$ is the infinitesimal distance vector. If $\delta\mathbf{r}$ lies in S, we must have $\delta\mathbf{q}\cdot\mathbf{h}=0$, *i.e.*, $\delta\mathbf{r}\cdot\nabla\mathbf{q}\cdot\mathbf{h}=0$. Hence $\nabla\mathbf{q}\cdot\mathbf{h}$ is parallel to $\mathbf{h}$ and we may write

$$\nabla\mathbf{q}\cdot\mathbf{h} = \lambda\mathbf{h}, \tag{2.1}$$

where λ is a scalar.

Now

$$\nabla\mathbf{q} = \mathbf{q}\nabla - \rho\times I, \tag{2.2}$$

where $\rho(=\text{curl }\mathbf{q})$ is the vorticity vector of the fluid and I is the unit dyadic (see (32) on p. 149 of [2]). Since $(\mathbf{q}\nabla)\cdot\mathbf{h}=\mathbf{h}\cdot\nabla\mathbf{q}$, (2.1) and (2.2) give

$$\mathbf{h}\cdot\nabla\mathbf{q} = \rho\times\mathbf{h} + \lambda\mathbf{h}. \tag{2.3}$$

The differential form of the equation of a stream line which meets S at P is

$$\delta\mathbf{r}\cdot\nabla\mathbf{q} = \mu\delta\mathbf{r}, \tag{2.4}$$

where μ is a scalar. The theorem now follows by comparing the last two equations. If $\delta\mathbf{r}$ is parallel to $\mathbf{h}$, then (2.4) holds with $\delta\mathbf{r}$ replaced by $\mathbf{h}$; hence $\lambda=\mu$ and $\rho\times\mathbf{h}=0$. Conversely, if $\rho\times\mathbf{h}=0$, then (2.3) shows that there is a stream line through P in the direction of $\mathbf{h}$.

3. The statement of the theorem has supposed the dividing stream line to be unique. Suppose that in addition to the stream line in the direction of $\mathbf{h}$ we

† This work was carried out at the Bristol Aeroplane Company.

have a stream line in the direction of $\delta\mathbf{r}^*$ which meets S obliquely at P. Let $\mathbf{t}$ be the unit vector along the projection of $\delta\mathbf{r}^*$ in the tangent plane to S at P, and let $\mathbf{b}=\mathbf{t}\times\mathbf{h}$.

From (2.2), (2.3) and (2.4), with $\rho\times\mathbf{h}=0$, we have

$$\lambda\mathbf{h}\cdot\delta\mathbf{r}^* = \mathbf{h}\cdot\nabla\mathbf{q}\cdot\delta\mathbf{r}^* = \delta\mathbf{r}^*\cdot\nabla\mathbf{q}\cdot\mathbf{h} = \mu\delta\mathbf{r}^*\cdot\mathbf{h} = \mu\mathbf{h}\cdot\delta\mathbf{r}^*.$$

Since $\delta\mathbf{r}^*$ does not lie in S, $\mathbf{h}\cdot\delta\mathbf{r}^*\neq 0$. Hence $\lambda=\mu$. That is, $\delta\mathbf{r}^*$ and $\mathbf{h}$ are two characteristic vectors of the dyadic $\nabla\mathbf{q}$ corresponding to the same characteristic number $\lambda=\mu$. It follows that $\mathbf{t}$ is also a characteristic vector of $\nabla\mathbf{q}$ with the same characteristic number, for $\mathbf{t}$ is parallel to $\delta\mathbf{r}^*-(\delta\mathbf{r}^*\cdot\mathbf{h})\mathbf{h}$. We thus have

$$\mathbf{t}\cdot\nabla\mathbf{q} = \lambda\mathbf{t}, \quad \text{and} \quad \mathbf{h}\cdot\nabla\mathbf{q} = \lambda\mathbf{h}. \tag{3.1}$$

Since $\rho\times\mathbf{h}=0$, we may write $\rho=\rho\mathbf{h}$. From (2.2), $\mathbf{t}\cdot\nabla\mathbf{q}=\mathbf{t}\cdot(\mathbf{q}\nabla)+\mathbf{t}\times\rho=\nabla\mathbf{q}\cdot\mathbf{t}+\rho\mathbf{b}$. Also, $\mathbf{h}\cdot\nabla\mathbf{q}=\nabla\mathbf{q}\cdot\mathbf{h}$. Thus (3.1) is equivalent to

$$\nabla\mathbf{q}\cdot\mathbf{t} = \lambda\mathbf{t} - \rho\mathbf{b}, \quad \text{and} \quad \nabla\mathbf{q}\cdot\mathbf{h} = \lambda\mathbf{h}.$$

Hence $(\mathbf{b}\cdot\nabla\mathbf{q}+\rho\mathbf{t})\cdot\mathbf{t}=(\mathbf{b}\cdot\nabla\mathbf{q}+\rho\mathbf{t})\cdot\mathbf{h}=0$; *i.e.*, $\mathbf{b}\cdot\nabla\mathbf{q}+\rho\mathbf{t}$ is parallel to $\mathbf{b}$ and we write

$$\mathbf{b}\cdot\nabla\mathbf{q} = \Lambda\mathbf{b} - \rho\mathbf{t}, \tag{3.2}$$

where Λ is a scalar.

The equations given by (3.1) and (3.2) determine the dyadic $\nabla\mathbf{q}$ since $\mathbf{t}$, $\mathbf{h}$ and $\mathbf{b}$ form an orthonormal set of vectors. We have

$$\begin{aligned}\nabla\mathbf{q} &= \lambda\mathbf{tt} + \lambda\mathbf{hh} + \Lambda\mathbf{bb} - \rho\mathbf{bt} \\ &= \lambda I + (\Lambda - \lambda)\mathbf{bb} - \rho\mathbf{bt}.\end{aligned}$$

This equation shows that if there is more than one (non-tangential) stream line through P, then all rays through P in the normal plane through any other determine directions of stream lines. Furthermore, if there are three non-coplanar (non-tangential) stream lines through P, then $\rho=0$ and $\Lambda=\lambda$, and any ray through P determines the direction of a stream line.

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A CONDITION FOR DIAGONABILITY OF MATRICES

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A square matrix is said to be *diagonable* if it is similar to a diagonal matrix. Various necessary and sufficient conditions for the diagonability of matrices are to be found in the literature. (See, for example, 1.) In this note we establish yet another such condition, which does not appear to have been stated previously.

By a *group matrix* we shall understand any matrix which is an element in

some multiplicative group of matrices. Every non-singular matrix is, of course, a group matrix; but groups of singular matrices also exist and were first investigated by Ranum (2). We shall prove the following

THEOREM. *A matrix A is diagonable if and only if $A-\omega I$ is a group matrix for every (complex) value of ω.*

All matrices considered below are assumed to be of type $n\times n$. We denote by I the unit matrix, by $R(A)$ the rank of A, and by diag $(\omega_1, \cdots, \omega_n)$ the diagonal matrix with diagonal elements $\omega_1, \cdots, \omega_n$. If $\mathfrak{V}$ is a vector space, then $d(\mathfrak{V})$ denotes its dimension.

A few preliminary results will be required.

LEMMA 1. *All matrices in a multiplicative matrix group have the same rank.*

Let A be an arbitrary element and U the unit element of a matrix group Γ. Denoting by B the inverse element of A, we have $U=AB$, $A=AU$. Therefore

$$R(U) = R(AB) \leqq R(A) = R(AU) \leqq R(U),$$

so that $R(A)=R(U)$ for all $A\in\Gamma$.

LEMMA 2. *If $R(AB)=R(B)$, then $R(ABC)=R(BC)$.*

Denote by $\mathfrak{N}_T$ the space of all vectors x such that $Tx=0$. Then

$$d(\mathfrak{N}_T) = n - R(T). \tag{1}$$

Hence, by hypothesis, $d(\mathfrak{N}_{AB})=d(\mathfrak{N}_B)$. But, trivially, $\mathfrak{N}_{AB}\supset\mathfrak{N}_B$, and so

$$\mathfrak{N}_{AB} = \mathfrak{N}_B. \tag{2}$$

Next, let $x\in\mathfrak{N}_{ABC}$, so that $ABCx=0$. Then $Cx\in\mathfrak{N}_{AB}$ and therefore, by (2), $Cx\in\mathfrak{N}_B$, *i.e.*, $BCx=0$. Thus $x\in\mathfrak{N}_{BC}$, and we have shown that $\mathfrak{N}_{ABC}\subset\mathfrak{N}_{BC}$. But, trivially, $\mathfrak{N}_{ABC}\supset\mathfrak{N}_{BC}$; and therefore $\mathfrak{N}_{ABC}=\mathfrak{N}_{BC}$. The assertion now follows by (1).

LEMMA 3. *If $\mu(x)$ is the minimum polynomial of A and $h(x)$ is a polynomial such that $h(A)=O$, then $\mu(x)$ divides $h(x)$.*

LEMMA 4. *If all zeros of the minimum polynomial of A are simple, then A is diagonable.**

Both these results are well-known.

Proof of the theorem. (i) Suppose that A is diagonable, and write $S^{-1}AS$ $=$diag $(\omega_1, \cdots, \omega_n)$. Then

$$S^{-1}(A - \omega I)S = \text{diag } (\omega_1 - \omega, \cdots, \omega_n - \omega). \tag{3}$$

Now the matrix on the right-hand side of (3) is an element in the matrix group Γ consisting of the matrices

* The converse statement is also true, but not necessary for our purpose.

$$\text{diag}\,(\epsilon_1 x_1, \cdots, \epsilon_n x_n),$$

where $x_1, \cdots, x_n$ take all non-zero (complex) values and ϵ_r is defined as 0 or 1 according as $\omega = \omega_r$ or $\omega \neq \omega_r$. Hence $A - \omega I$ is an element of the group $S\Gamma S^{-1}$, and so is a group matrix.

(ii) Suppose, next, that $A - \omega I$ is a group matrix for every value of ω. Then, by Lemma 1,

$$R(A - \omega I) = R\{(A - \omega I)^2\} \qquad \text{(all } \omega\text{)}. \tag{4}$$

Now let ω be any zero of the minimum polynomial $\mu(x)$ of A, so that $\mu(x) = (x-\omega)^k \phi(x)$, where $k \geqq 1$ and $\phi(\omega) \neq 0$. There exist polynomials $f(x)$ and $g(x)$ such that

$$(x - \omega)^k f(x) + \phi(x) g(x) = 1. \tag{5}$$

This implies that

$$(A - \omega I)^k f(A) + \phi(A) g(A) = I. \tag{6}$$

Writing $E = \phi(A) g(A)$, multiplying both sides of (6) by E, and using the fact that $\mu(A) = O$, we obtain $E^2 = E$. Putting $Z = (A - \omega I)E$, we therefore have

$$Z^m = (A - \omega I)^m E, \tag{7}$$

so that

$$Z^k = O. \tag{8}$$

In view of (4), (7), and Lemma 2, we at once obtain $R(Z) = R(Z^2)$. Hence, by Lemma 2 and (8),

$$R(Z) = R(Z^2) = \cdots = R(Z^k) = 0,$$

so that $Z = O$, *i.e.*,

$$(A - \omega I)\phi(A) g(A) = O.$$

It follows by Lemma 3 that $(x-\omega)^k \phi(x)$ divides $(x-\omega)\phi(x)g(x)$. But, by (5), $g(\omega) \neq 0$ and therefore $k = 1$. All zeros of $\mu(x)$ are therefore simple; and by Lemma 4, A is diagonable.

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1. M. P. Drazin, On diagonable and normal matrices, Quart. J. Math. (Oxford) (2), vol. 2, 1951, pp. 189–198.

2. A. Ranum, The group-membership of singular matrices, Amer. J. Math., vol. 31, 1909, pp. 18–41.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

ON THE REPRESENTATION OF POWERS OF PRIMES BY POLYNOMIALS WITH INTEGRAL ZEROS

J. G. CAMPBELL, University of Kentucky

Let $f(x)$ represent a polynomial with integral coefficients. Let $(r_1, r_2, \cdots, r_j)$ represent the total set of integral zeros of $f(x)$ and note that $f(x)$ may have additional non-integral zeros. If $f(v)=p^a$, where p is a prime and a is an integer, then the integral value v is definitely restricted to a finite set of integers, dependent on the r's. Let r_i, $1 \leqq i \leqq j$, be any integral zero of $f(x)$. Then $f(x)=(r_i-x)q(x)$ and $f(m)=(r_i-m)q(m)$, where m is any integer not a zero of $f(x)$. Designate the total set of positive and negative divisors of $f(m)$ by the set $(d_1, d_2, \cdots, d_k)$. But (r_i-m) is one of the divisors of $f(m)$ and $(r_i-m)+m=r_i$. Hence, r_i is in the set $(d_1+m, d_2+m, \cdots, d_k+m)$. We use this fact to prove the following theorem.

THEOREM 1. *Let $(g_1, g_2, \cdots, g_n)$ represent the total set of zeros of $f(x)$. Let the subset $(r_1, r_2, \cdots, r_j)$, $j \geqq 1$, represent all integral zeros of $f(x)$. If $f(v)=p^a$, where p is a prime and a is a positive integer, then $v=r_s \pm p^t$, $s=1, 2, \cdots, j$, $t=0, 1, 2, \cdots, a$, for at least one s and one t.*

Proof. Finding a value v for which $f(v)=p^a$ is equivalent to finding an integral zero v of $F(x)=f(x)-p^a$. It is clear that $m=r_s$, $s=1, 2, \cdots, j$, is not a zero of $F(x)$ for any s, since $F(m)=-p^a$. The total set of divisors of $F(m)$ is $(1, p, p^2, \cdots, p^a, -1, -p^2, \cdots, -p^a)$. Hence, the zeros of $F(x)$ are included in the set $(r_s \pm p^t)$, $s=1, 2, \cdots, j$, $t=0, 1, 2, \cdots, a$. This completes the proof of the theorem.

It follows immediately that $f(v)$ represents a composite number for all integral $v \neq r_s \pm p^t$, $s=1, 2, \cdots, j$, $t=0, 1$. It is possible in certain cases to restrict the value v to a single integer if the polynomial $f(x)$ has only one integral zero. $F(x)=x^n-k^n-p^b$ is an example. Here n is an odd prime, the integer b is less than or equal to n, and k is a known integer. The only integral zero of x^n-k^n is k. The integral zeros of $F(x)=x^n-k^n-p^b$ are in the set $(k \pm p^t)$, $t=0, 1, 2, \cdots, b$. If $v^n=k^n+p^b$, then the inequalities $k<v<k+p$ hold. We must take the positive sign for p^t in the above set, since v is greater than k. The only positive value in the set for which $v<k+p$ is $k+1$. Hence, $v=k+1$. The following theorem is immediate.

THEOREM 2. *Let $x^n-y^n=p^b$, where n is an odd prime, p is a prime, and $b \leqq n$. Then x and y are consecutive integers.*

Equations of this type actually occur, *e.g.*, $8^3-7^3=13^2$.

ON "SOLVING" DIFFERENTIAL EQUATIONS

R. C. Buck, University of Wisconsin

Most texts in differential equations regard a problem as solved when the solution of a differential equation is reduced to the evaluation of one or more integrals. In many cases, the form in which such an answer appears gives little indication of the qualitative nature of the solutions. The purpose of this note is to point out that the mean value theorem can sometimes be applied to obtain quite precise information; this serves as an additonal "practical" motivation for the introduction of these tools into the engineering-type of calculus course.

Let us consider the first order linear equation

$$\frac{dy}{dx} - 2xy = x^2, \qquad x \geqq 0. \tag{1}$$

By the usual procedure, we immediately obtain the general solution

$$\begin{aligned} y &= e^{x^2}\int_0^x t^2 e^{-t^2}dt + Ce^{x^2} \\ &= e^{x^2}\left\{\int_0^x t^2 e^{-t^2}dt + C\right\}. \end{aligned} \tag{2}$$

However, the student would have considerable difficulty in visualizing the behavior of the solution curves from this. The following is certainly more useful.

Theorem. *The general solution of* (1) *is*

$$y = Y_0(x) + Ae^{x^2} \tag{3}$$

where the particular solution Y_0 *obeys the inequalities*

$$-\frac{x}{2} \geqq Y_0(x) \geqq -\frac{x}{2} - \frac{1}{4x}, \qquad x \geqq 2. \tag{4}$$

Corollary. *The equation* (1) *has precisely one solution (namely,* $y = Y_0(x)$*) for which* dy/dx *is bounded, as* $x \to \infty$.

Proof. Examining the solution (2) we notice that the improper integral $\int_0^\infty t^2e^{-t^2}dt$ converges. Let its value be K. [The fact that K may be evaluated precisely, with $K = \sqrt{\pi}/4$, is not relevant to our argument.] Then

$$K = \int_0^x t^2e^{-t^2}dt + \int_x^\infty t^2e^{-t^2}dt,$$

so that (2) may be written:

$$y = e^{x^2}\left\{K + C - \int_x^\infty t^2e^{-t^2}dt\right\}.$$

Choosing $C=-K$, we obtain as a particular solution of (1),

$$y = Y_0(x) = -e^{x^2}\int_x^\infty t^2e^{-t^2}dt.$$

To estimate the rate of growth of $Y_0(x)$ as x increases, we write this as

$$Y_0(x) = \frac{-\int_x^\infty t^2e^{-t^2}dt}{e^{-x^2}} = \frac{F(x)}{G(x)}. \tag{5}$$

Observing that $\lim_{x\to\infty} F(x)=\lim_{x\to\infty} G(x)=0$, we apply the mean value theorem, and conclude that there is a point $\bar{x}>x$ such that $Y_0(x)=F'(\bar{x})/G'(\bar{x})$. However, $F'(x)=x^2e^{-x^2}$ and $G'(x)=-2xe^{-x^2}$, so that $Y_0(x)=-\frac{1}{2}\bar{x}$. Since $\bar{x}>x$, we have shown that $Y_0(x)<-\frac{1}{2}x$ for any $x>0$.

Consider now

$$Y_0(x)+\frac{1}{2}x = -e^{x^2}\int_x^\infty t^2e^{-t^2}dt+\frac{1}{2}x$$

$$= \frac{-\int_x^\infty t^2e^{-t^2}dt+\frac{1}{2}xe^{-x^2}}{e^{-x^2}} = \frac{f(x)}{g(x)}.$$

Applying the same method, we have $Y_0(x)+\frac{1}{2}x=f'(\bar{x})/g'(\bar{x})$ for some $\bar{x}>x$. This time, $f'(x)=\frac{1}{2}e^{-x^2}$ and $g'(x)=-2xe^{-x^2}$ so that $Y_0(x)+\frac{1}{2}x=-1/(4\bar{x})$. Since $\bar{x}>x$, this gives $Y_0(x)+\frac{1}{2}x>-1/(4x)$. Together, these inequalities give formula (4) and the theorem is proved.

We observe that what has been shown is that $Y_0(x)=-\frac{1}{2}x+O(x^{-1})$, with an estimate on the error. To prove the corollary, we return to the original differential equation (1) and, putting $y=Y_0(x)$, we have

$$\frac{dy}{dx} = 2xy+x^2 = 2x\left\{-\frac{1}{2}x+O(x^{-1})\right\}+x^2$$

$$= -x^2+O(1)+x^2 = O(1).$$

Thus, the solution Y_0 has a bounded derivative. To see that this is the only one, we observe that

$$\frac{d}{dx}\left\{Y_0(x)+Ae^{x^2}\right\} = Y_0'(x)+2Axe^{x^2},$$

which can never be bounded as x increases, unless $A=0$.

It is worth pointing out that this process can be carried out further to yield additional terms, arriving at (for example)

$$Y_0(x) = -\frac{1}{2}x - \frac{1}{4}x^{-1} + \frac{1}{8}x^{-3} - \frac{3}{16}x^{-5} + O(x^{-7}).$$

This suggests that one should seek a series representation for Y_0 of this form; one easily obtains the formal expansion

$$Y_0(x) = -\frac{1}{2}x + \sum_{k=0}^{\infty}(-1)^{k-1}\frac{(2k)!}{k!}\frac{x^{-2k-1}}{4^{k+1}}.$$

However, this series is divergent for every x; the student is now ready for the general notion of asymptotic series and their application.

TWO METHODS FOR FINDING THE ANGLE OF ROTATION

O. J. KARST, Stevens Institute of Technology

In eliminating the xy term from the general second degree equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ by rotation of axes, the rotation angle θ is given by the formula, $\tan 2\theta = B/(A-C)$. In order to obtain the $\sin\theta$ and $\cos\theta$ to use in the rotation transformation equations

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta,$$

the half angle formulas are generally used.

There are two other methods which are equally effective and are based on more familiar elementary methods than the half angle formulas. The first method is geometrical and is most easily shown by an example.

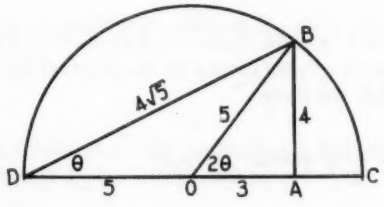


FIG. 1

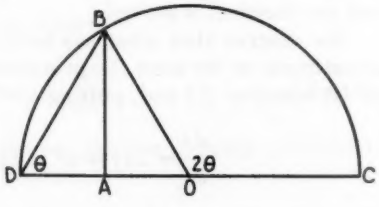


FIG. 2

Let $\tan 2\theta = 4/3$ (see Figure 1). Then construct 2θ as the central angle in a semi-circle, whose radius will then be 5. Since 2θ is measured by arc BC, angle BDC will be measured by $\frac{1}{2}$ arc BC and hence angle $BDC=\theta$. Hence from triangle DAB we obtain

$$\sin\theta = \frac{1}{\sqrt{5}} \quad \text{and} \quad \cos\theta = \frac{2}{\sqrt{5}}.$$

If $\tan 2\theta$ is negative, the figure is essentially the same as shown in Figure 2.

The second method is analytic. Let

$$\tan 2\theta = \frac{B}{A - C} = K$$

then

$$\frac{2 \tan \theta}{1 - \tan^2 \theta} = K$$

or

$$K \tan^2 \theta + 2 \tan \theta - K = 0.$$

Hence $\tan \theta = (-1 \pm \sqrt{1+K^2})/K$ from the quadratic formula. The positive root gives the usual acute positive angle of rotation. The negative root leads to a negative angle of rotation which is the numerical complement of the first angle. Either one will eliminate the xy term from the original equation.

GRAPHICAL CONSTRUCTION OF LOGARITHMIC SPIRALS

DIRAN SARAFYAN, Lamar State College of Technology

Let $A_1A_2A_3$ and $A_2A_3A_4$ be any two right triangles having in common the side A_2A_3 (Fig. 1). By drawing straight lines, departing from A_4, parallel to $\overline{A_1A_2}$ and $\overline{A_2A_3}$ consecutively, we obtain a graph which we shall call a rectangular spiral.

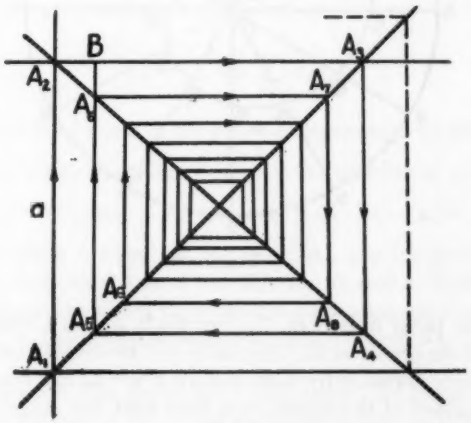


FIG. 1

Let $\overline{A_1A_2} = a$ and the ratio $A_3A_4/A_1A_2 = r$. Then $\overline{A_3A_4} = ar$.

Considering that the triangles $A_1A_2A_3$, $A_3A_4A_5$, $A_5A_6A_7$ *etc.* are similar, it can be easily seen that the magnitudes of the opposite sides of a rectangular spiral form the geometric progression

$$\overline{A_1A_2} = a \qquad \overline{A_3A_4} = ar \qquad \overline{A_5A_6} = ar^2 \qquad \overline{A_7A_8} = ar^3 \cdots .$$

Thus, the following theorem can be announced:

THEOREM: *The magnitudes of the opposite sides of a rectangular spiral form a geometric progression.*

Through the use of this theorem, the logarithmic spiral, the exponential and the logarithmic curves and the damped sinusoidal waves can be traced. We shall construct only the logarithmic spiral since the construction of the others follows more or less the same pattern.

Let the logarithmic spiral be represented by

$$\rho = a_0 a^{-\theta} \qquad 0 < a \neq 1.$$

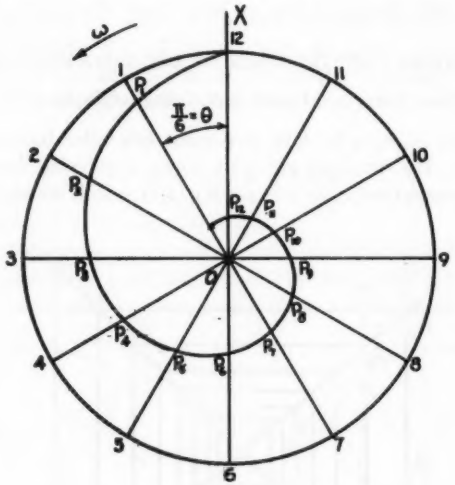


FIG. 2

Let OX be the polar axis (Fig. 2). For $\theta = 0$, $\rho = a_0$. Consider a circle with radius a_0 and with its center at O. This circle will be divided into n equal sectors, where n is an integer divisible by 4. In Figure 2, $n = 12$ has been chosen.

From the equation of the spiral, it is seen that the magnitude of the polar vectors OP_i with $i = 0, 1, 2, \cdots$, form a geometric progression of common ratio $a^{-\pi/6}$.

If on Figure 1, we take $\overline{A_1A_2} = a_0$ and $\overline{A_3A_4} = a_0a^{-\pi/6}$, then in accordance with the previous theorem, the vertical sides of the rectangular spiral will be equal in length to the magnitude of polar vectors OP_i which will enable us to plot the points $P_1, P_2, P_3, \cdots$, of the logarithmic spiral.

If we consider the sequence of points P_i by successive groups of three points through which we draw an arc of a circle we obtain a graph which will almost coincide with a logarithmic spiral. However when great accuracy is not essential, one can as well join the sequence of points P_i by a smooth curve.

References

1. Irmgard Flugge-Lotz, Discontinuous Automatic Control, Princeton University Press, Princeton, New Jersey, 1953, pp. 163–65.

2. Timoshenko and Young, Vibration Problems in Engineering, Van Nostrand Co., New York, Third Edition, 1955, pp. 72–73.

THE NUMBER OF COMPONENTS OF CERTAIN GRAPHS

W. R. Utz, University of Missouri

Let $s, \alpha_1, \alpha_2, \cdots, \alpha_n$ be positive integers and let $a_1 > a_2 > \cdots > a_n$ be reals. The graph of

$$y^s = (x - a_1)^{\alpha_1}(x - a_2)^{\alpha_2} \cdots (x - a_n)^{\alpha_n} \tag{1}$$

is obviously connected if s is odd or if $n = 1$. Define, for each positive integer p,

$$g(p) = \begin{cases} 0, & \text{if } p \text{ is even;} \\ 1, & \text{if } p \text{ is odd.} \end{cases}$$

We shall prove the following theorem.

THEOREM. *If s is even and $n > 1$, the graph of* (1) *has*

$$1 + \sum_{i=1}^{n-1} g\left(\sum_{j=1}^{i} \alpha_j\right) \tag{2}$$

components (each isolated point of the graph being counted as one component).

The theorem is easily generalized to give the number of components of

$$y^s = G(x)(x - a_1)^{\alpha_1}(x - a_2)^{\alpha_2} \cdots (x - a_n)^{\alpha_n},$$

wherein s is a positive integer, $\alpha_1, \alpha_2, \cdots, \alpha_n$, are integers (not necessarily positive) and $G(x)$ is defined and of one sign for all real x. Besides its interest in an algebra or analytic geometry class the theorem answers a natural topological question. Similar, but generally less elementary, theorems are found, for example, in [1]

Proof of the theorem. The proof is by induction. Let $n = 2$. If α_1 is odd, it is easily verified that the graph of (1) has two components (one of which is a point in case α_2 is even). If α_1 is even, the curve is easily seen to be connected.

Before proceeding to the second step in the induction we observe that since

$$(x - a_{r+1})^{\alpha_{r+1}} > 0 \quad \text{for} \quad x > a_{r+1},$$

it is clear that the components of

$$y^s = (x - a_1)^{\alpha_1} \cdots (x - a_r)^{\alpha_r} \tag{3}$$

and

$$y^s = (x - a_1)^{\alpha_1} \cdots (x - a_r)^{\alpha_r}(x - a_{r+1})^{\alpha_{r+1}} \tag{4}$$

will be the same in number for $x > a_{r+1}$.

Suppose that (2) holds for $n = 2, 3, \cdots, r$.

Case 1. $\alpha_1+\alpha_2+ \cdots +\alpha_{r-1}$ is odd. In this case $(x-a_1)^{\alpha_1} \cdots (x-a_{r-1})^{\alpha_{r-1}} < 0$ for $x < a_{r-1}$. Thus, if α_r is odd the components of (3) and (4) are the same regardless of whether α_{r+1} is even or odd and, since α_r is odd, the added term in (2) is 0. Similarly, if α_r is even, (4) has one more component (being a point if α_{r+1} is even) than (3) regardless of whether α_{r+1} is even or odd and, since α_r is even, the added term, $g(\alpha_1+\alpha_2+ \cdots +\alpha_r)$, in (2) is 1.

Case 2. $\alpha_1+\alpha_2+ \cdots +\alpha_{r-1}$ is even. In this case

$$(x - a_1)^{\alpha_1} \cdots (x - a_{r-1})^{\alpha_{r-1}} > 0 \qquad \text{for } x < a_{r-1}.$$

If α_r is odd, (4) has one more component than (3) has regardless of whether α_{r+1} is even or odd and (2) is increased by 1 since $\alpha_1+ \cdots +\alpha_r$ is odd. If α_r is even, (4) and (3) must have the same number of components and, since $\alpha_1+ \cdots +\alpha_r$ is even, the added term in (2) is 0. This completes the proof of the theorem.

Reference

1. Wilfred Kaplan, The topological structure of a variety defined by an equation, Compositio Mathematica, vol. 5, 1938, pp. 327–346.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1221. *Proposed by S. W. Golomb, University of Oslo, Norway*

If n is an integer greater than 1, then 2^n-1 is never a perfect square, cube, or higher power.

E 1222. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If we designate by C_1, C_2, C_3 the sides of the regular convex heptagon and of the two regular star heptagons inscribed in a circle of radius R, then

$$C_1^2 + C_2^2 + C_3^2 = 7R^2.$$

E 1223. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Starting with two arbitrary non-negative numbers a_1 and a_2, build a sequence a_k, $k=1, 2, 3, \cdots$, by taking alternately the arithmetic and the geometric means of the two preceding numbers, for example,

$$0, 1, 1/2, \sqrt{2}/2, (1+\sqrt{2})/4, \cdots.$$

Evaluate $\lim a_k$ as $k \to \infty$.

E 1224. *Proposed by G. K. Wenceslas, Providence, R. I.*

Let G be a finite group of rigid motions (or more generally affine transformations) of a Euclidean space. Then there is a point of the space which is left fixed by all the transformations of G.

E 1225. *Proposed by L. R. Ford, Illinois Institute of Technology*

Find necessary and sufficient conditions that it be possible to lay an $a \times b$ rug on an $A \times B$ floor. (Take $a \geqq b$, $A \geqq B$.)

SOLUTIONS

Bingo Cards

E 1188 [1955, 655]. *Proposed by Alice Anderson, Student, East Tennessee State College*

Each card used in the game of Bingo is made up of a square array of 25 numbers. The five numbers in the first column are selected from 1 to 15, those in the second column from 16 to 30, those in the third column from 31 to 45, those in the fourth column from 46 to 60, and those in the fifth column from 61 to 75. The space common to the third column and third row is left blank. If no card can have a row, column, or diagonal identical to a row, column, or diagonal of another card, how many different cards can be made?

Solution by A. C. Goddard and F. I. John, Dugway, Utah. The number of different cards certainly cannot exceed the number of cards with different middle columns. This is, in fact, the total number of possible different cards since the number of cards with different rows and diagonals is greater than this number $C(15, 4) = 1365$.

Also solved by G. B. Charlesworth and A. R. Hyde.

TRIGGonometry Exam

E 1191 [1955, 727]. *Proposed by C. W. Trigg, Los Angeles City College*

Identify the mathematical terms corresponding to the following unorthodox definitions: (1) an ice-cream container, (2) dull, stupid, (3) the forefinger, (4) a carpenter's tool, (5) a commission merchant, (6) having a nice discernment, (7) a writing instrument, (8) to yoke, (9) to turn up the earth with a snout, (10) one who does not conform to the standards of the gang, (11) to court in Colonial times, (12) a small stream, (13) a mercenary in the American Revolution, (14) to increase in strength and influence, (15) a prescribed movement of a fleet, (16) a somite or metamere, (17) the first entrance of any house in the calculation of nativities, (18) an attribute attached to the predicate narrowing its extent, (19) a genus of pulmonate land snails, (20) a group of two coupled feet, (21) a sandpiper which breeds in Arctic regions.

Contributions by H. W. Becker, George Cherlin, A. R. Hyde, M. S. Klamkin, D. C. B. Marsh, C. S. Ogilvy, Walter Penney, C. F. Pinzka, Azriel Rosenfeld, Samuel Wolf, and the proposer.

(1) cone, cylinder, isoperimeter, quartic
(2) dense, obtuse, obvious, subnormal, trivial
(3) digit, index
(4) brace, level, line (plumb), plane, right angle, rule, scale, square
(5) differential operator, factor
(6) acute, differentiating, discriminant
(7) pencil
(8) adjoin, combine, connect, couple, join
(9) root
(10) eccentric, deviate, non-conformal, odd number, radical, singular, square, variant
(11) bundle, determine the bending moment, meet
(12) branch
(13) Hessian
(14) augment, expand, exponentiate, monotonically increase, multiply
(15) echelon, evolution, isodrome, 180° turn (German fleet maneuver in the battle of Jutland), operation, tac (tack)
(16) homologous part, segment
(17) cusp, first derived set, first element of an ordered set, perfect number (ask the man who owns one), prime, sign
(18) characteristic, condition, determinant, kernel, limit
(19) helix, limaçon, slug
(20) base, basis, diameter (dimeter), dyad, measure, metric, pair, pedal pair, ten binary digits, syzygy
(21) knot

Late solutions by Vern Hoggatt and J. B. Muskat.

Much Ado About Nothing

E 1192 [1955, 728]. *Proposed by Edgar Karst, Independence, Missouri*

Let AB, BC be two adjacent sides of a regular nonagon inscribed in a circle of center O. Let M be the midpoint of AB and N the midpoint of the radius perpendicular to BC. Show that angle $OMN = 30°$. (Attention Professor Umbugio.)

Editorial Note. Professor Umbugio need not involve himself in a lengthy geometric or trigonometric argument, for, since O, A, M, N are concyclic, angle OMN = angle $OAN = 30°$.

Also solved by W. A. Al-Salam, J. L. Baker, Leon Bankoff, A. P. Boblétt, G. B. Charlesworth, A. E. Danese, J. E. D'Atri, Hüseyin Demir, J. M. Elkin, Edward Fleisher, H. E. Gould, Bernard Greenspan, W. T. Gregorzak, Emil Grosswald, Vern Hoggatt, J. P. Hoyt, Raymond Huck, A. R. Hyde, P. W. M. John, P. B. Kantor, M. S. Klamkin, Sam Kravitz, M. A. Laframboise, Elmer Latshaw, B. R. Leeds, D. C. B. Marsh, A. R. Nolstad, C. S. Ogilvy, Margaret Olmsted, Walter Penney, P. W. A. Raine, A. P. Rhodes, L. A. Ringenberg, D. A. Robinson, Azriel Rosenfeld, J. W. Ross, C. M. Sandwick, Sr., J. P. Scholz, E. P. Starke, Sister M. Stephanie, J. A. Tierney, Chih-yi Wang, Alan Wayne, Hazel S. Wilson, Samuel Wolf, David Zeitlin, and the proposer. Late solutions by Louisa R. Alger, G. W. Day, Roger Hou, F. I. John, Josef Langr, C. F. Pinzka, and A. I. Thaler.

Two Problems Connected with Rocket Research

E 1193 [1955, 728]. *Proposed by L. C. Barrett, University of Utah, and Herbert Knothe, Holloman Air Development Center*

(1) An ellipse has one focus at the center of two concentric circles, is tangent internally to the larger circle, and has at least one point in common with the smaller circle. Find the length of the maximum major axis of all such ellipses.

(2) Let the inner of two concentric circles represent a homogeneous spherical mass and the outer a prescribed orbit. Determine the range of specific energy values a particle may be given if it is to traverse a plane elliptical path from the surface of the sphere and just reach, but not cross, the circular orbit.

Solution by the Proposers. (1) Denote the radii of the circles by R and $R+h$, and let a typical ellipse have major axis $2a$, minor axis $2b$, and focal distance c. We may then write the polar equation of the ellipse as

$$r = b^2/(a - c\cos\phi).$$

For the ellipse to intersect the inner circle, it must be possible to determine ϕ's for which the equation

$$R = (R + h)(2a - R - h)/[a - (R + h - a)\cos\phi]$$

is satisfied. Solving this for $\cos\phi$ we obtain

$$\cos\phi = [(R+h)^2 - (R+2h)a]/R(R+h-a).$$

As a increases, $\cos\phi$ decreases so that the maximum a occurs when $\cos\phi = -1$. Using the last equation, we find that the maximum value of $2a$ is $2R+h$, or the sum of the radii of the two circles.

(2) Since the particle moves in an inverse square force field, one focus of its elliptical path is at the center of the circles. Let R and $R+h$ be their radii and k the gravitational constant. Then the specific energy E of a particle traversing an orbit of major axis $2a$ is

$$E = -kM/2a.$$

The required range of specific energies is thus determined by the maximum and minimum values $2a$ can assume. These, from part (1), are $2R+h$ and $R+h$, respectively. If $2a = R+h$, the ellipse degenerates into a straight line segment. Hence

$$-kM/(R+h) < E \leqq kM/(2R+h).$$

Also solved by D. C. B. Marsh (partially), J. W. Ross, and R. H. Wilson, Jr.

The Proposers stated that these problems arose in connection with one phase of rocket research at the Holloman Air Development Center.

Construction of a Triangle

E 1194 [1955, 728]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Construct a triangle ABC given A, $ma+b$, $na+c$, where m and n are given positive integers. (Dedicated to Professor W. B. Carver, the geometer of precision.)

Solution by R. L. Caskey, Oklahoma A. and M. College. If we construct triangle ADE with $AD = na+c$, $AE = ma+b$, angle $DAE = A$, we note that the problem becomes that of drawing a secant meeting the sides AD, AE in B, C, respectively, such that $DB:BC:CE = n:1:m$.

To solve this problem we construct, as follows, a quadrilateral $DAHI$ homothetic to the required quadrilateral $DBCE$ and having D as center of homothety. On EA lay off $EF = (m/n)AD$. Draw FG parallel to DE. Draw circle $(A, AD/n)$, cutting line FG in H. Draw HI parallel to FE, cutting DE in I.

We now obtain quadrilateral $DBCE$ by drawing DH, letting it intersect AE in C, and then drawing CB parallel to HA. If, in this construction, B falls between A and D and C between A and E, then triangle ABC is a solution of the problem. For, by construction, $DA:AH:HI = 1:1/n:m/n = n:1:m$. Therefore $DB:BC:CE = n:1:m$, and thus $DA = DB+BA = na+c$ and $EA = EC+CA = ma+b$.

Examples can be produced showing two, one, or no solutions. The number of solutions is controlled by the fact that circle $(A, AD/n)$ may cut line FG in two, one, or no points, and also by the fact that for the construction to be acceptable, B must fall between A and D and C between A and E.

We may remove the restriction that m and n be positive integers by letting m and n be represented by line segments and by letting the unit segment be given.

Also solved by W. B. Carver, Roger Hou, A. R. Hyde, Edgar Karst, M. S. Klamkin, Beckham Martin (three ways), D. C. B. Marsh, H. D. Ruderman, C. M. Sandwick, Sr., and the proposer. Late solutions by Vern Hoggatt and Josef Langr.

For the Proposer's treatment of this problem see *Mathesis*, vol. LXIV (1955), nos. 6–7–8, p. 286, art. 42.

Carver communicated a lengthy and remarkably complete algebraic treatment, which more than upholds the description of him in the dedication. The interested reader might like to carry out Caskey's construction on the following numerical examples given by Carver. The examples exhibit the possibility of two solutions and show how the construction may break down in various ways to give one or no solution.

$5a + b = 3$	$3a + b = 16$	$3a + b = 4$	$5a + b = 3$
$2a + c = 2$	$2a + c = 7$	$2a + c = 7$	$2a + c = 2$
$\cos A = -1/2$	$\cos A = 1/2$	$\sin A = 1/5$	$\cos A = 19/20$

$5a + b = 10$	$5a + b = 3$	$5a + b = 3$	$9a + b = 2$
$4a + c = 9$	$2a + c = 2$	$4a + c = 5$	$2a + c = 1$
$\cos A = 1/2$	$\cos A = 43/45$	$\cos A = 99/100$	$\cos A = -1/2$

$2a + b = 5$	$3a + b = 5$	$9a + b = 2$
$2a + c = 4$	$3a + c = 4$	$2a + c = 1$
$\cos A = 197/208$	$\cos A = 17/18$	$\cos A = (2 + \sqrt{504})/25$

A Conditional Inequality

E 1195 [1955, 728]. *Proposed by G. E. Bardwell, University of Denver*

If $n = 2, 3, 4, \cdots$, and m is fixed and positive, for what positive values of p less than 1 is $\ln n < mn^p$?

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. If $mn^p > \ln n$, then

$$p > \ln [(\ln n)/m]/\ln n = (\ln x - \ln m)/x,$$

where $x = \ln n$. Now the maximum value of $(\ln x - \ln m)/x$ occurs when $x = em$ and is equal to $1/em$. Thus

$1 > p > 1/em$ provided $m > 1/e$.

Also solved by G. B. Charlesworth, A. E. Danese, Nathaniel Grossman, A. R. Hyde, Lawrence Isenecker, D. C. B. Marsh, Walter Penney, E. P. Starke, Chih-yi Wang, David Zeitlin, and the proposer. Late solutions by A. J. Goldman and Roger Hou.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4693. *Proposed by J. V. Whittaker, University of California, Los Angeles*

Show that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{m!n!}{(m+n+2)!}=\frac{\pi^2}{6}.$$

4694. *Proposed by R. W. Bagley, University of Kentucky*

There are simple examples which show that an uncountable topological space in which limits are unique (hence T_1) is not necessarily Hausdorff. Are there such examples for countable spaces? Here "limit" is used in the usual sense of limit of a sequence where the directed set is the positive integers rather than limit of a generalized sequence as defined by Kelley and others. With this general definition (where the directed set is allowed to vary), Kelley proved that a space is Hausdorff if, and only if, limits are unique. (See, Convergence in Topology, *Duke Mathematical Journal*, (1950), pp. 277–283.)

4695. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that if in a cyclic quadrangle the Simson line of one vertex with respect to the triangle formed by the other three is perpendicular to the Euler line of that triangle, then the same property holds for the other vertices of the quadrangle.

4696. *Proposed by T. M. Apostol, California Institute of Technology*

Let $\Lambda(n)$ denote Mangoldt's function, defined as follows:

$$\Lambda(n) = \begin{cases} \log p \text{ if } n \text{ is a power of the prime } p \\ 0 \text{ otherwise.} \end{cases}$$

For $n>1$, let $f_m(n)$ denote the number of representations of n as a product of m factors, where each factor is greater than 1 and the order of the factors is taken into consideration. (*E.g.*, $f_2(24)=6$ and $f_3(16)=3$.) Prove the identity

$$\Lambda(n) = (\log n) \sum_{m=1}^{\nu(n)} \frac{(-1)^{m+1}}{m} f_m(n), \qquad (n > 1),$$

where $\nu(n)$ is the total number of prime factors of n, *i.e.*, $\nu(n)=a_1+a_2+\cdots$ if $n=p_1^{a_1}p_2^{a_2}\cdots$.

4697. *Proposed by Leonard Carlitz, Duke University*

Find the roots of the equation

$$y^n - ny^{n-2} + \frac{n(n-3)}{2!}y^{n-4} + \cdots + (-1)^r \frac{n(n-r-1)\cdots(n-2r+1)}{r!}y^{n-2r} + \cdots = 2a,$$

in particular when $a=1$.

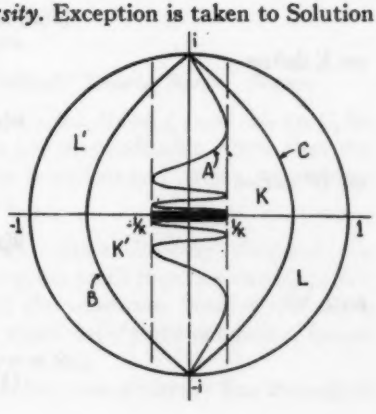
SOLUTIONS

Zero of a Complex Function

4594 [1954, 427]. *Proposed by Edgar Reich, Rand Corporation, Santa Monica, California*

If $f(z)$ is a complex-valued continuous function of the complex variable z, such that $f(z)=z$ whenever $|z|=1$, show that $f(z)$ has at least one zero in $|z|<1$.

IV. *Note by S. L. Gulden, Lehigh University.* Exception is taken to Solution I [1955, 591]. In this solution if one sets $f(z)=u(z)+iv(z)$ it is required that the set $\{z \mid u(z)=0\}$ contain a path joining i to $-i$. The following shows that this need not be true. (See figure).



Let

$$D = \{z \mid |z| \leqq 1\},$$

$$E = \left\{ z \,|\, z = \frac{1}{k} \sin \frac{\pi}{|y|} + iy;\ 0 < y \leqq 1 \right\},$$

$$F = \left\{ z \,|\, z = x;\ -\frac{1}{k} \leqq x \leqq \frac{1}{k},\ k > 100 \right\}.$$

Let $A = E \cup F$. Let B and C be circular arcs which do not meet A save at i and $-i$ (by choosing k sufficiently large this can be arranged) and assume them parametrized by y, *i.e.*, B is given by $x = x_1(y)$, C by $x = x_2(y)$, $-1 \leqq y \leqq 1$.

Let

$$G = \left\{ z \,|\, z = x + iy,\ \frac{1}{k} \sin \frac{\pi}{|y|} \leqq x \leqq x_2(y),\ y \neq 0 \right\},$$

$$H = \left\{ z \,|\, z = x,\ -\frac{1}{k} \leqq x \leqq x_2(0) \right\},$$

$$K = G \cup H,$$

$$G' = \left\{ z \,|\, z = x + iy,\ x_1(y) \leqq x \leqq \frac{1}{k} \sin \frac{\pi}{|y|},\ y \neq 0 \right\},$$

$$H' = \left\{ z \,|\, z = x,\ x_2(0) \leqq x \leqq \frac{1}{k} \right\},$$

$$K' = G' \cup H',$$

$$L = \left\{ z \,|\, z = x + iy,\ x_2(y) \leqq x \leqq (1 - y^2)^{1/2} \right\},$$

$$L' = \left\{ z \,|\, z = x + iy,\ -(1 - y^2)^{1/2} \leqq x \leqq x_1(y) \right\}.$$

The sets K, K', L, L' are all closed.

Define $u(z)$ on D as follows: Let d be the usual metric on D,

$$(d(z_1, z_2) = |z_1 - z_2|);$$

on K define

$$u(z) = \frac{d(z, A)}{5},$$

on K' define

$$u(z) = -\frac{d(z, A)}{5};$$

on L let

$$t = \frac{x - x_2(y)}{(1 - y^2)^{1/2} - x_2(y)}$$

and define

$$u(i) = u(-i) = 0,$$

$$u(z) = (1 - t)\frac{d(x_2(y) + iy, A)}{5} + t(1 - y^2)^{1/2}, \qquad z \neq i, -i;$$

on L' let

$$t = \frac{x_1(y) - x}{x_1(y) + (1 - y^2)^{1/2}}$$

and define

$$u(i) = u(-i) = 0,$$

$$u(z) = (1 - t)\left[-\frac{d(x_1(y) + iy, A)}{5}\right] - t(1 - y^2)^{1/2}, \qquad z \neq i, -i.$$

The following facts are not difficult to establish:

1. $u(z)$, thus defined on D, is continuous.
2. $\{z \mid u(z) = 0\} = A$.
3. There is no path in A joining i to $-i$.
4. For $|z| = 1$, $u(z) = \mathrm{Re}(z)$.

If one now takes for v the function defined by $v(z) = \mathrm{Im}(z)$ and for f the function $f(z) = u(z) + iv(z)$, where u is as defined above, then the argument of Solution I fails to show that $f(z)$ has a zero in D.

It should be noted, however, that although the supporting argument does not hold, the theorem: *Under the given hypotheses, for any z_1 with $|z_1| < 1$ there is a $z_2 \in D$ such that $f(z_2) = z_1$*, is nevertheless true and can be proved by a simple generalization of solution II of the problem.

Confocal Conics

4630 [1955, 127]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Having given two confocal conics C_1 and C_2, let M_1 on C_1 and M_2 on C_2 be so chosen that the tangents at these points are perpendicular. Show that the envelope of M_1M_2 is another conic having the same foci and having asymptotes passing through the intersections of C_1 and C_2.

Solution by E. J. F. Primrose, University College, Leicester, England. We assume that the geometry is complex. If M_1 is given on C_1 there are two diametrically opposite points M_2 on C_2 which satisfy the condition. Similarly, if M_2 is given on C_2 there are two points M_1 on C_1 which satisfy the condition. Hence there is a (2, 2) correspondence between M_1 and M_2.

Now let O be a general point of the plane. Since an arbitrary line through O

meets C_1 in two points M_1, and C_2 in two points M_2, there is a (4, 4) correspondence between OM_1 and OM_2. Such a correspondence has eight double lines, so there are eight lines M_1M_2 which pass through a general point O. The class of the envelope M_1M_2 is therefore eight.

Now let the common points of C_1 and C_2 be Q, R, S, T, where Q and S are a diametrically opposite pair, and so are R and T. Then Q as a point of C_1 corresponds to Q as a point of C_2, since confocal conics cut orthogonally. Hence OQ is one of the double lines, and so are OR, OS, OT. It follows that part of the envelope consists of the four points Q, R, S, T. The rest of the envelope is of class four.

Of the four common tangents of C_1 and C_2, two pass through I and two through J, where I and J are the circular points at infinity. Let one common tangent touch C_1 at N_1 and C_2 at N_2. Then N_1 corresponds to N_2, since a line through a circular point is self-perpendicular. Also, if O is a point on the tangent, N_1 counts twice as an intersection of ON_1N_2 with C_1. Hence N_1N_2 counts twice as a line of the envelope, and so do the other three common tangents. Also, Q on C_1 corresponds to S on C_2 and S on C_1 corresponds to Q on C_2, so QS, quite apart from belonging to the pencil with vertex Q and to the pencil with vertex S, counts twice as a line of the envelope: so does RT.

Now if six lines are double lines of an envelope of class four, either

(i) the six lines are the sides of a quadrangle, so the envelope consists of four points, or

(ii) the envelope is a repeated conic.

Clearly (i) is not the case, so the envelope is a repeated conic. Since it touches the four common tangents of C_1 and C_2, it is confocal with C_1 and C_2. Since it touches QS and RT, and these are two lines through the center, they must be the asymptotes.

It is worth noting that, since every tangent to the envelope counts doubly, if M_1M_2 meets C_1 again at P_1 and C_2 again at P_2, the tangents at P_1 and P_2 must also be perpendicular. Incidentally the dual argument would show that the locus of the point of intersection of the perpendicular tangents consists of the four common tangents and the circle $QRST$ counted twice: hence the other tangents from the point of intersection are also perpendicular.

The problem is solved in Baker, *Principles of Geometry*, vol. II, pp. 127–133, but the proof is long and difficult, since the problem is treated as a particular case of a remarkable general theorem. A proof may also be found in C. Smith, *Conic Sections*, p. 311. Smith does not mention the asymptote property but this is easily proved on the basis of the given analysis.

Also solved by Josef Langr, E. J. F. Primrose (second solution), Sister M. Stephanie, Chih-yi Wang, and the Proposer. The Proposer points out that the proposition of the problem is the reciprocal with respect to L of the following considerably easier proposition: Being given two circles C_1 and C_2 and points M_1 and M_2 on C_1 and C_2 respectively, so chosen that angle $M_1LM_2=\pi/2$, where

L is one of the limit points of the pencil of circles determined by C_1 and C_2, then the tangents to C_1 and C_2 at M_1 and M_2 intersect on a circle belonging to the pencil of circles.

A Permutation Problem

4633 [1955, 258]. *Proposed by W. E. Weissblum, Republic Aviation Corporation, Farmingdale, N. Y.*

An even number, n, of chairs are arranged in a circle and numbered 1 through n. An equal number of people are numbered 1 through n and line up in an arbitrary manner. Prove that they can be seated in such a way as to preserve their order and such that no person's number is the same as that of his chair.

Solution by John Isbell, Aberdeen Proving Ground, Maryland. The permutation f of the persons makes seating without coincidence impossible if and only if for each j there is an i such that $f(i)+j\equiv i \pmod{n}$. For each i there is exactly one j, and hence $i\to j$ is a permutation g.

Thus we have a problem in the ring of integers modulo n. Now if n is odd there is a permutation $f(i)=g(i)=i/2$ such that $f(i)+g(i)=i$. But if n is even, the sum of all elements of the ring is $1+2+\cdots+n\equiv n/2$. Thus the sum of all $f(i)$ is n, and all $g(i)$ is n, and $i\to f(i)+g(i)$ is not a permutation.

Also solved by W. J. Blundon, Marshall Freimer and S. W. Golomb, A. J. Goldman, R. E. Greenwood, J. E. Householder, Free Jamison, D. C. B. Marsh, Katsuyoshi Memezawa, Leo Moser, D. J. Newman, J. V. Pennington, James Singer, and Michael Skalskyj.

Elliptic Integrals

4634 [1955, 258]. *Proposed by Chih-yi Wang, University of Minnesota*

Let a real number k, $0<k<1$, be given. Define the elliptic integrals

$$A_s=\int_0^{\pi/2}\sin^{2s}t\sqrt{1-k^2\sin^2t}\,dt, \qquad B_s=\int_0^{\pi/2}\frac{\sin^{2s}t\,dt}{\sqrt{1-k^2\sin^2t}}$$

for $s=0, 1, 2, \cdots$. Prove that

$$(a)\ B_r=\frac{B_{r-1}-A_{r-1}}{k^2}, \qquad (b)\ A_r=\frac{(2rk^2-1)A_{r-1}+(1-k^2)B_{r-1}}{(2r+1)k^2},$$

for $r=1, 2, 3, \cdots$.

Solution by T. F. Mulcrone, S. J., St. Charles College, Grand Coteau, La. Part (a) follows immediately upon substitution of the values for A_{r-1} and B_{r-1} in terms of the defining integrals.

For part (b), we have

$$A_{r-1} - A_r = \frac{1}{2r-1}\int_0^{\pi/2} \cos t\sqrt{1 - k^2 \sin^2 t}\cdot d(\sin^{2r-1} t),$$

and an integration by parts gives

$$A_{r-1} - A_r = \frac{1}{2r-1}\left[A_r + k^2(B_r - B_{r+1})\right],$$

from which part (b) follows upon two applications of (a).

Also solved by W. J. Blundon, L. Carlitz, P. L. Chessin, G. B. Findley, V. D. Gokhale, Peter Henrici, H. F. Hunter, A. R. Hyde, M. S. Klamkin, G. N. Lance, W. R. McEwen, D. C. B. Marsh, R. E. Messick, Kovina Milosevich, C. D. Olds, B. E. Rhoades, P. G. Rooney, M. R. Spiegel, O. E. Stanaitis, J. A. Tierney, and the Proposer.

The Inversion in an Additive Group

4635 [1955, 259]. *Proposed by A. E. Babbitt, Jr., Rutgers University*

Let G be an additive Abelian group which has no element of order 2 and let f be a one to one mapping of G onto G. Then f is the inversion (*i.e.*, $f(x) = -x$) if and only if the following two conditions are satisfied:

I. The neutral element 0 is the only element of G invariant under f.

II. For every element $a \in G$, there exists an element $b = b(a)$ in G such that, for all $x \in G$,

$$f(x - a) = f^{-1}(x + b) - a.$$

Solution by W. R. Scott, University of Kansas. The hypothesis that G be commutative can be dropped provided that the equation in II is replaced by

II′. $$f(x - a) = -a + f^{-1}(x + b).$$

If $f(x) = -x$, then I is obvious; and II′ is satisfied with $b = -2a$. Conversely, assume I and II′ are true. Then $f(0) = 0$, and letting $x = a$ in II′ we get $a = f^{-1}(a + b)$ or

$$b = -a + f(a).$$

Again, with $x = 2a$, II′ yields $a + f(a) = f^{-1}(2a - a + f(a))$, or

$$f(a + f(a)) = a + f(a).$$

Hence, by I, $f(a) = -a$.

Also solved by J. D. Baum and Angelo Margaris, B. J. Boyer, R. H. Breusch, T. A. Brown, R. G. Crook and Velma R. Vogler, David Ellis, M. P. Epstein, V. D. Gokhale, A. J. Goldman, C. D. Gorman, D. S. Greenstein, Virginia S. Hanly, E. H. Hietbrink, Douglas Holdridge, S. P. Hughart, H. H. Johnson, W. G. Leavitt, H. D. Lipsich, D. C. B. Marsh, Katuyosi Memezawa, R. H. Oehmke, F. P. Palermo, B. E. Rhoades, P. G. Rooney, H. D. Ruderman,

P. T. Schaefer, Richard Scoville, James Singer, R. E. Swett, G. H. M. Thomas, Olga Taussky-Todd and John Todd, C. J. Vanderlin, Chihi-yi Wang, J. V. Whittaker, H. H. Wicke, F. J. Witt, Donald Ylvisaker, and the Proposer.

Editorial Note. This result is equivalent to a result obtained by de Kerekjarto, *C. R. Acad. Sci.*, Paris, 210, 1940, pp. 288–9. It was generalized to arbitrary groups in a paper by Taussky and Todd, Inversion in groups, *Quarterly Journal of Mathematics*, 12 (1941), pp. 65–7. Ellis adds the following references: *Anzeiger der Akad. der Wissenschaften* (Wien), 1930, 140–2; *Math. Zeitschr.*, 33, 1931, 412; *Publicationes Math.* (Debrecen), 2, 1951, 1–25.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio

The Bequest of the Greeks. By Tobias Dantzig. Charles Scribner's Sons, New York, 1955. 191 pages. $3.95.

Ever since the appearance of Professor Dantzig's famous book *Number: The Language of Science*, other works by this gifted interpreter of the world of mathematics have been eagerly purchased and read by both the professional mathematician and the layman. In this new book, which is the first volume of a projected trilogy, Professor Dantzig again takes up his favorite study of the evolution of mathematical ideas. Here he is concerned with those "problems, principles and procedures which modern mathematics has inherited from Greek antiquity."

The book is divided into two parts. Part One, entitled "The Stage and the Cast," is designed for the general reader and makes practically no demands upon the reader's technical skill in mathematics. Part Two, entitled "In Retrospect," is an anthology of the Greek bequest, and is definitely more technical in character.

Part One opens with three chapters devoted to the historical mystery of the Greek achievement, to the founders of deductive mathematics, and to the origins of geometry. Here we find one of those relatively rare instances in which Professor Dantzig foregoes his general reluctance to offer his personal speculations on historically uncertain matters, and in it Thales emerges considerably stronger, and Pythagoras considerably weaker, than in traditional accounts of this period. Next follows a pair of interesting chapters devoted to Thales' measurement of an Egyptian pyramid and to the occult developments of the

golden ratio. If one is to consider seriously the alleged feat of Thales in measuring the height of a pyramid, one must then look into some of the practical difficulties involved, such as the determination of the distance from the center of the floor of the pyramid to the apex of the pyramid's shadow. These matters have received little attention in most accounts of Thales' accomplishment, and Professor Dantzig explains an historically plausible conjecture as to how Thales may have circumvented the difficulties. The chapter on occult fringes of mathematics carries the mystical golden section from the early Greeks up through the schizophrenic writings of Johann Kepler. Part One concludes with a chapter called "The Pseudomath," which many of us read when it appeared separately in *The Scientific American*, and one entitled "The Interdiction," devoted to the Greek decree that geometers limit themselves in their constructions to the straightedge and compass; the subsequent history of mathematics owes much to this deep-rooted prohibition.

The anthology of Part Two is made up of eight topics—the Pythagorean theorem, Pythagorean triples, the lunes of Hippocrates, the quadratrix of Hippias, the Euclidean algorithm, Archimedes' quadrature of the parabola, Hero's formula for the area of a triangle, and Greek trigonometry. Each of these issues has survived the passage of time, and in some form or other is still alive today. Some of the items have only recently reached full fruition, while others still remain largely incomplete. Professor Dantzig handles these items with consummate skill and interest. For example, the chapter on Pythagorean triples carries the story from the Greek period, through the work of Fibonacci in the Middle Ages, to some of the associated problems of modern times to which partial answers, solutions, and supplements have been given by Fermat, Euler, Lagrange, Legendre, and Gauss. One's conclusion is that, after having engaged the attention of top mathematicians for twenty-five hundred years, the Pythagorean equation cannot yet be said to be exhaustively solved. As regards the Pythagorean triples, Professor Dantzig strangely says nothing about the very probable ancient Babylonian treatment as substantiated by the remarkable tablet *Plimpton 322*. To the present reviewer, the high spot of Part Two is the chapter devoted to the lunes of Hippocrates; the recent work on these lunes described by Professor Dantzig is not too widely known.

Professor Dantzig's anthology contains only eight topics. There are certainly other items that come to mind, such as the Eudoxian theory of proportion and its culmination in modern times in the Dedekind theory of irrational numbers, and Euclid's parallel postulate and the subsequent liberation of geometry and modern development of axiomatics. But, as appears in a statement in the Epilogue to the book, Professor Dantzig intends to deal with other items in the next volume of the projected trilogy. We all certainly look forward to the forthcoming volumes of the trilogy.

There is much that is scholarly in Professor Dantzig's book, and there are many instances of the author's skilful use of the written word. It seems a pity to have to add a few words of a derogatory nature. But these derogatory

comments are almost all of a superficial sort. It would seem inexcusable that a work which otherwise is so well executed should have been so poorly proofread. One of the greatest sins in this respect is the frequent and confusing disagreement between the lettering of a figure and the description of the figure in the accompanying text. This fault is found, for example, in Figures 8*a* and 8*b* on p. 62, Figure 24 on p. 140, Figure 27 on p. 161, and Figure 29*a* on p. 171. On p. 63 we find "*witch'es foot*"; on p. 67 some quotation marks are omitted; on p. 95 we find the bad hyphenation "quadr-"; in the last line on p. 100 we find $ACUUV$ instead of $ACUV$; on p. 102 we find a' in Figure 15 where a should appear; on p. 130, line 15, the u should be a v, and the last displayed equation on this page should read "$u:v=a:b$"; on p. 132, line 5, we should read "(3,2)" for "(3.2)", and similarly on this same page in line 1 of section 10 we should read "(3,1)" for "(3.1)"; in section 10 on p. 132 we should interchange u and v; on p. 133 we again find the incorrect "(3.1)" and "(3.2)", and this same oversight is perpetuated on pp. 134, 135, and 136; on p. 145 there is a word missing in line 2 of section 1; the first sentence of section 2 on p. 146 is at fault; on p. 155 c is *not* always a better approximation than either a or b; on p. 156 equation (13.6) should commence with u_{n+1}; on p. 162 the definition of Heronian triples is carelessly stated; on p. 163, line 3 from the bottom, we should read "Figure 28" for "Figure 31"; on p. 164, lines 8 and 9, the language "it is greater, equal or less than 1" is faulty; on p. 166, line 3, we should read "$3y^3$" for "$3y$"; on p. 174 Figure 31*a* should be labelled Figure 31*b*, and Figure 31*b* should be labelled Figure 31*a*; on p. 175, line 5, we should read "$\cos \frac{1}{2}\theta$" instead of "$\cos \theta$," and the same correction should also be made in the accompanying Figure 32. Finally, mathematical equations are parts of sentences, and should therefore be properly punctuated. There are many omissions of such expected punctuation in this book. See, for example, pp. 126, 130, 141, 144, 156, 157.

HOWARD EVES
University of Maine

Principles of Mathematics. By. C. B. Allendoerfer and C. O. Oakley. New York, McGraw-Hill Book Company, 1955. xv+448 pages. $5.00.*

In their new book *Principles of Mathematics*, Professors Allendoerfer and Oakley have made a distinct contribution to the undergraduate curriculum. There is almost universal awareness among college mathematics teachers that something fundamental is happening to our beginning courses and there is now more widespread concern over these courses than at any time in the past fifty years. This concern is reflected in the outpouring of genuinely new textbooks.

* Due to the interest in this book, the Book Review Editor felt that two reviews were warranted, and takes full responsibility for both. E.P.V.

The editor-in-chief is both pleased and embarrassed by the attention shown this book. He has followed his usual policy of giving a free hand to Professor Vance and consequently is printing both reviews. C.B.A.

Unfortunately, not all of these new approaches are good, and it may be quite a few years before we again reach any basic stability in the subject matter for our beginning courses.

These new texts run the gamut from complete rigor to merely talking about mathematics, and again from much to almost no concern for the role of mathematics in the outside world. Until very recently there has been a distinct tendency for those texts which are most concerned with applications to be least concerned with rigor and inversely. The present book represents an exception to this tendency; the authors have used applications to illustrate the need for careful formulation of concepts. They have proceeded throughout with heuristic discussion which should help the student realize the need for rigor, and to understand what is happening as well as giving some insight into the origin of definitions and proofs. Although their book leans toward the rigorous side of the range, they have kept in mind the needs and capabilities of the American college freshman. Their treatment is completely honest, *i.e.*, they give careful definitions, accurate statements of theorems, and when proofs are omitted the student is told what is happening and is usually referred to a proof elsewhere; the student who goes on to higher mathematics will not have to "unlearn" what he has been taught in his first year.

One interesting feature is the inclusion of some applications in social science areas. Fortunately from the point of view of those (including the reviewer) who believe that some set theory is important in a beginning course, the social sciences provide a fertile field for illustration of set concepts.

This text was originally designed for a year course for freshmen at a liberal arts college, and it assumes that the student enters with a working knowledge of intermediate algebra. However, there is enough material and the arrangement is such that the text can be used for a variety of courses, several of which are outlined in the preface.

In terms of subject matter this is not a really radical text. The new subject matter comes in Chapter 1, Logic; parts of Chapter 2, The Number System; Chapter 3, Groups; much of Chapter 4, Fields; Chapter 5, Sets and Boolean Algebra; and Chapter 13, Statistics and Probability. However, most of the traditional topics in college algebra, trigonometry, and many topics from analytic geometry can be found here, albeit sometimes in abbreviated form. Moreover, the authors take full advantage of the new viewpoints of the earlier chapters in their development of this classical material.

The exposition is uniformly good. The authors have searched for the best formulation of each topic. Especially noteworthy is Chapter 8 which presents what has been called "the unavoidable essentials of trigonometry" in a very elegant and concise manner.

The exercises make good reading; but their adequacy can only be checked by classroom test. There may not be enough exercises for alternation from term to term. Answers are given to selected exercises.

Some special features of text are: a more than usual use of mathematical

abbreviations ($\forall$, $\exists$, *etc.*) together with a list of symbols at the beginning of the book; a brief introduction to congruences; application of Boolean algebra to electric networks; inclusion of polynomial calculus; a brief bibliography at the end of each chapter.

The text is unusually accurate. The following minor errors were noted: p. 131, in discussing absolute value the word "positive" should be replaced by "non-negative"; p. 149, it should be mentioned that section 8 is limited to relations on the real numbers; p. 192, in the definition of periodic function one should specify $a>0$; p. 294, the footnote seems unnecessary.

The user of this text has to face the question of what follows it in the second year. The answer for now is a standard course in analytic geometry and calculus. However, it is to be hoped that the authors will carry on with their task and provide a sequel of equal stature and merit and which is not shackled by the chains of tradition.

R. M. THRALL
University of Michigan

In the preface to this book the authors reveal that they were motivated by two convictions: first, that "large parts of the standard undergraduate curriculum in mathematics are obsolete"; and second, that "some of the content and much of the spirit of modern mathematics can be incorporated in courses given to our beginning students." A rough idea of the content which this text tries to incorporate into freshman mathematics is most quickly gained by a glance at the thirteen chapter headings: Logic, The Number System, Groups, Fields, Sets and Boolean Algebra, Functions, Algebraic Functions, Trigonometric Functions, Exponential and Logarithmic Functions, Analytic Geometry, Limits, The Calculus, Statistics and Probability. The authors suggest that a year course meeting three times a week might reasonably expect to cover the first ten chapters. The spirit of modern mathematics is evident in the precise definitions, the explicit recognition of undefined terms, the clear distinction between axioms and theorems, and the meticulous care with which most of the theorems are proved. Complex numbers are defined in terms of ordered pairs of real numbers; the function concept is introduced as a set of ordered pairs, no two of which have the same first element; and the subject of limits gets the full ϵ, δ treatment. Even though the level of rigor and abstraction is unusually high for freshman texts, the style is smooth and pleasantly informal. Interesting historical references help to give the subject that 3-D look which is so deplorably lacking in the usual introductory text. References at the end of each chapter are to suggest outside reading for the student, but most of them will probably be of much greater help to the teacher. For example, *A Survey of Modern Algebra* by Birkhoff and MacLane is a favorite reference, recommended after several of the chapters, but this reviewer does not get many beginning students who could profitably read from this excellent—for senior math majors—book.

Generally speaking, the supply of problems is good for the more novel subjects, but very meager indeed for such standard topics as factoring, fractions, and quadratic equations. The subject of fractions is treated in a brief section of about a page and a half in the chapter on fields. For that considerable body of freshmen who have not mastered the arithmetic of rational numbers, this treatment is grotesquely inadequate. The usual "word problems" dealing with mixtures, rates, levers, investment, *etc.*, are conspicuously missing. These omissions give a clue as to which parts of the standard curriculum the authors consider to be obsolete. It will come as a surprise and a disappointment to many that matrices are not even mentioned, even though elementary matrix theory is simpler than some of the material included, and a knowledge of it would suggest very helpful examples in the discussion of abstract mathematical systems.

A limited edition of a preliminary form of this book came out in 1953, and this reviewer enthusiastically adopted it as the text in one of his freshman classes. It should be pointed out that this class was certainly not above average in ability and it is possible that the selection process used might have made it slightly inferior. This fact might mitigate some of the following criticism. Except for containing a more exhaustive treatment of the number system in the second chapter, that early version was much the same as the one being reviewed. Both are naturally much more interesting to the teacher than the usual text with its greater emphasis on drill in the traditional topics, and this makes one hope very much that the students will also prefer the new approach. The reviewer's hopes along this line were completely shattered. It is possible of course that more skillful teaching might have avoided this disillusionment, but the conclusion reached here is that the laudable goal which the authors set for themselves is simply too ambitious for the great majority of beginning students. The learning process of the beginner seems to move most naturally from the particular to the general and from the concrete to the abstract. To a large extent, this text tries to lead the student in the opposite direction. For example, in the sixth chapter the student is presented with the modern concepts of "function" and "relation" in all their abstract generality. It is not until later, in Chapters VII, VIII and IX, that he studies the rational, algebraic, trigonometric, logarithmic and exponential functions. It is the belief of this reviewer that the same ground can be covered more successfully, though probably not in one year, if the student is first made to feel quite familiar with the elementary functions. The more abstract question of precisely what it is that all the objects which we call functions have in common becomes meaningful, as the history of mathematics plainly shows, only after one has accumulated considerable experience with many particular types of functions.

The conceptual framework with which the student is presented is so spacious that his meager items of knowledge are insufficient to support the vast generalizations in any meaningful way. The result of this frequently seems to be that the text is giving elegant answers to questions which the student is not asking and

which he can scarcely believe are significant. This makes the problem of motivation extremely difficult.

The criticisms expressed here naturally suggest the question of whether or not this text might be suitable for carefully selected groups of freshmen who already know what used to be considered as high school mathematics. Though not wanting to deny that possibilities along this line may exist, this reviewer is strong in the belief that much of the contents of this text should be deferred till a junior-senior course based perhaps on the well-known text of Birkhoff and MacLane. In preparation for such a course, the elements of matrix theory could be included in freshman algebra and analytic geometry.

W. R. MANN
University of North Carolina

Topological Transformation Groups. By D. Montgomery and L. Zippin. New York, Interscience Publishers, Inc., 1955. 282 pages.

The book under review is a carefully written study of topological groups giving special emphasis to locally compact topological groups and transformation groups. It is substantially more readable than most books dealing with topics of roughly the same level of abstraction and should, therefore, be very useful to all kinds of mathematicians, from first year graduate students to mature mathematicians whose primary interest is in some other branch of mathematics.

The book consists of three parts. The first part (chapter I) introduces the reader to the basic topological and group theoretical notions. Topological groups, local groups, transformation groups, manifolds, as well as covering spaces are defined in this first chapter. It is one of the pleasant features of the book that whenever any of the basic concepts is used for the first time in a new context in a later chapter, a reference is made to the appropriate page of the first chapter.

The second part of the book (chapters II to IV) is primarily concerned with locally compact groups and their relation to Lie groups. Besides reviewing the more important classical work in the field, these chapters summarize the important work which has recently been done on Hilbert's fifth problem, work which until now could be found in papers only. It is best to state this problem in the authors' own words (page 70). Before doing this we note that a transformation group is said to act effectively on a manifold, M, if the identity element of the group is the only element mapping each point of M into itself. The authors, then, discuss the key question of this section:

"Let us now consider the following questions, the second and third of which are asked by Hilbert:

If a locally compact group acts effectively on a manifold M then

(1) is G necessarily locally euclidean,

(2) if the group G is locally euclidean, is it a Lie group in some appropriate coordinate,

(3) if G is a Lie group can coordinates be chosen in G and M so that the transformation functions are analytic?

The answer to (2) · · · is yes · · · . The answer to (1) is unknown except in some special cases · · · . The answer to (3) is no."

In the process of developing these theories use is made of the work of Gleason and Yamabe in proving the equivalence of the concepts of a Lie group and of a locally compact group without small subgroups (a group is said to have no small subgroups if there exists a neighborhood of the identity which contains no subgroups except the identity itself). Finally, the question of approximating compact and locally compact groups by Lie groups is extensively investigated in these chapters.

The last part of the book (chapters V and VI) presupposes a certain amount of background in algebraic topology. On the other hand, it is probably the most rewarding part from the point of view of the mathematician who is primarily interested in the still unsolved problems of mathematics. In this last part of the book, the authors discuss transformation groups deriving some and merely stating some of the other results while clearly indicating some of the existing gaps. The following is a partial list of the topics which are treated: differentiability properties of the transformations, conjugacy of subgroups of a Lie group, orbits and cross sections (the orbit of a point is the totality of its images under the transformation group G, a cross section is a 1:1 image of the orbits in the neighborhood of some point) periodic and pointwise periodic transformations, and finally an analysis of the action of a compact connected group on 3-space.

Summarizing, this is a very interesting book from just about every point of view. The organization and motivation are unusually good. Most potential readers can look forward to an enjoyable experience.

F. Haas
University of Connecticut

Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley and Los Angeles, 1956. Twenty-five per cent discount to members of the Mathematical Association of America.

Volume I. *Contributions to the Theory of Statistics*

Contributors: Joseph Berkson; Z. W. Birnbaum; Herman Chernoff and Herman Rubin; Aryeh Dvoretzky; Sylvain Ehrenfeld; G. Elfving; Ulf Grenander and Murray Rosenblatt; J. L. Hodges, Jr. and E. L. Lehmann; Wassily Hoeffding; Samuel Karlin; L. Le Cam; Herbert Robbins; Murray Rosenblatt; Charles Stein; B. L. Van der Waerden.

Cloth. List Price $6.00. Publication Date: August 18, 1956.

VOLUME II. *Contributions to Probability Theory*

Contributors: David Blackwell; Salomon Bochner; K. L. Chung; A. H. Copeland, Sr.; J. L. Doob; Robert Fortet; J. M. Hammersley; T. E. Harris; Kiyosi Itô; Paul Lévy; Michel Loève; Eugene Lukacs; Karl Menger; Edith Mourier; R. Salem and A. Zygmund.

Cloth. List Price $6.50. Publication Date: October 10, 1956.

VOLUME III. *Contributions to Astronomy and Physics*

Contributors: Olin J. Eggen: Jesse L. Greenstein; Harold L. Johnson; Gerald E. Kron; Bengt Strömgren; G. C. McVittie; Jerzy Neyman; Elizabeth L. Scott and C. D. Shane; F. Zwicky; André Blanc-Lapierre and Albert Tortrat; M. Kac; J. Kampé de Fériet; Elliott Montroll; Norbert Wiener.

Cloth. List Price $6.25. Publication Date: June 1, 1956.

VOLUME IV. *Contributions to Biology and Problems of Health*

Contributors: James F. Crow and Motoo Kimura; Everett R. Dempster; Jerzy Neyman, Thomas Park and Elizabeth L. Scott; M. S. Bartlett; A. T. Bharucha-Reid; Chin L. Chiang, J. L. Hodges, Jr. and J. Yerushalmy; Jerome Cornfield; David Kendall; William F. Taylor.

Cloth. List Price $5.75. Publication Date: June 5, 1956.

VOLUME V. *Contributions to Econometrics, Industrial Research, and Psychometry*

Contributors: Kenneth J. Arrow and Leonid Hurwicz; Edward W. Barankin; C. West Churchman; Patrick Suppes; Albert H. Bowker; Cuthbert Daniel; Milton Sobel; T. W. Anderson and Herman Rubin; Frederick Mosteller; Herbert Solomon.

Cloth. List Price $5.75. Publication Date: June 30, 1956.

By a special arrangement, the members of the Association may purchase any or all of the five volumes at a 25 percent discount. To obtain this discount, orders must be sent to the Statistical Laboratory, University of California, Berkeley 4, California. They must refer to membership in the Association and be accompanied by checks for the correct amount. The checks should be made out to the University of California Press.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

ANNUAL MEETING OF ASEE

The annual meeting of the American Society for Engineering Education will be held on June 25–29, 1956 at Iowa State College, Ames, Iowa. The following program will be presented by the Mathematics Division of the Society.

Joint Conference with Iowa Section of the Mathematical Association of America, June 25, 1956, 2 P.M.: *Analog Computers and Research in Mathematics*, J. W. Cell, North Carolina State College; *Statistics in Engineering Education and Research*, Bernard Ostle, Montana State College; *Worktext Teaching Aid in Advanced Applied Mathematics*, G. I. Cohn, Illinois Institute of Technology; *Colleges, High Schools, and the Mathematics Curriculum*, Henry Van Engen, Iowa State Teachers College; *Mathematical Problems in Industry*, Morris Ostrofsky, Westinghouse Electric Company; *The Industrial Research Mathematician*, S. L. Levy, Midwest Research Institute.

Luncheon and Business Meeting, June 26, 1956, 12 Noon: *Engineering for Mathematicians*, C. O. Oakley, Haverford College (retiring Chairman of the Division).

Conference on Trends in Applied Mathematics, June 26, 1956, 2 P.M.: *Introductory Remarks* by Chairman, F. A. Ficken, University of Tennessee; *Fluid Mechanics*, M. H. Martin, University of Maryland; *Electro-magnetic Theory* (speaker to be announced); *Numerical Methods*, A. S. Householder, Oak Ridge National Laboratory; *Elasticity*, (speaker to be announced).

Joint Conference with Engineering Drawing, June 27, 1956, 2 P.M.: *Operational Symbolism for Graphical Processes*, S. A. Coons, Massachusetts Institute of Technology; *Graphical Field Mapping*, J. F. Calvert, University of Pittsburgh; *Graphical Analogs of Mathematical Processes*, J. F. Twigg, Massachusetts Institute of Technology.

Joint Conference with Industrial Engineering Division on Operations Research, June 28, 1956, 2 P.M.: *Development of Operations Research*, D. W. Stebbins, Iowa State College; *Tentative General Principles of the Design of Air Defense Systems*, Richard Blythe, Air Defense Command, U.S.A.F.; *Determination of Optimum Reserve Generating Capacity in an Electric Utility Industry*, E. L. Arnoff, Case Institute of Technology; *Engineering Applications on Queuing Theory*, L. D. Findley and T. A. Mortenson, Midwest Research Institute.

ASSOCIATION FOR COMPUTING MACHINERY

The annual meeting of the Association for Computing Machinery will be held at the University of California, Westwood Campus, Los Angeles, August 27–29, 1956. For information write G. W. King, Box 3251, Olympic Station, Beverly Hills, California.

CONFERENCE ON APPLIED MATHEMATICS

The University of Buffalo will hold a Conference on Applied Mathematics during the two-week period from June 11 to June 22, 1956. The principal lecturer will be Professor R. V. Churchill of the University of Michigan. The conference is sponsored by the Union Carbide and Carbon Corporation in order to further promote the use of mathematical methods in industry in the Niagara Frontier.

INSTITUTE OF NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS

The eighth annual Institute for Teachers and Professors of Mathematics, sponsored by the Association of Teachers of Mathematics in New England, will be held at Williams College, Williamstown, Massachusetts, August 16–23, 1956. Mrs. Frances L. Allen of Newton High School, Massachusetts, is General Chairman.

The program consists of morning and evening lectures and three periods daily for discussion groups (four choices each period) with topics of interest to teachers from the seventh grade up. Social hours and other recreational activities are also planned.

Some of the topics to be discussed are *Enrichment of the Teaching of Junior High School Mathematics*, *How to Integrate Algebra, Geometry, and Trigonometry with Conventional Textbooks*, *Creative Imagination in the Teaching of Mathematics*, *Number Systems*. In addition to the junior and senior high school laboratories, Professor E. A. Hoadley will be back with both beginners' and advanced courses in *Geometry Applied to Artistic Design*.

For complete program and other information, please write to Miss Barbara B. Betts, 138 Norfolk Avenue, Swampscott, Massachusetts.

ILLINOIS JOURNAL OF MATHEMATICS

The University of Illinois announces the forthcoming publication of the Illinois Journal of Mathematics, a quarterly devoted to the publication of basic research in pure and applied mathematics. The first number will appear early in 1957, and it is planned that each number will contain about 150 pages.

The present board of editors consists of Reinhold Baer, J. L. Doob, and A. H. Taub, University of Illinois; George W. Whitehead, Massachusetts Institute of Technology; and Oscar Zariski, Harvard University.

The Illinois Journal will publish papers in English, French, German, and Italian. Manuscripts should be sent to the Illinois Journal of Mathematics,

Department of Mathematics, University of Illinois, Urbana, Illinois, or to one of the editors.

The Illinois Journal of Mathematics is made possible by a grant from the bequest to the University of Illinois by the late Professor G. A. Miller, and by an initial subvention from the American Mathematical Society.

The subscription price is $9.00 a volume (four numbers), with a special rate of $5.00 for individual members of the American Mathematical Society. Subscriptions should be sent to the University Press, 207 Administration East, University of Illinois, Urbana, Illinois, U.S.A.

PERSONAL ITEMS

Lebanon Valley College announces: Associate Professor B. H. Bissinger has been elected to the status of member in the London Mathematical Society; also, the College has inaugurated, in the Harrisburg area, a curriculum in Operations Research designed for management personnel as well as engineers and scientists. The following courses have been given: Operations research for management; Linear programming and game theory in industry.

Dr. S. I. Askovitz, consultant in mathematics and staff physician at the Albert Einstein Medical Center and at the Wills Eye Hospital, Philadelphia, Pennsylvania, has been appointed Chief of the Tumor Registry at the Hospital of the University of Pennsylvania, as biostatistician in the field of cancer.

Mr. Souren Babikian has been appointed to an instructorship at Los Angeles City College.

Dr. R. W. Bagley, formerly a senior aerophysics engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas, has accepted a position as associate research scientist on the applied mathematics staff of Lockheed Aircraft Corporation, Van Nuys, California.

Mr. C. J. Biggerstaff, previously of the Flight Computer Group, Firestone Company, Los Angeles, California, has a position as an electronics engineer at the Research and Development Laboratories, Rheem Manufacturing Company, Downey, California.

Associate Professor Leonard Bristow of Wisconsin State College, Oshkosh, has been promoted to a professorship.

Professor Harold Chatland, Dean of the College of Arts and Sciences of Montana State University, is now Acting Dean of the Faculty.

Mr. M. M. Chirico, previously a mathematician at the Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland, has a position as a scientist for the Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania.

Associate Professor C. F. Christ of Johns Hopkins University has been appointed Associate Professor of Economics at the University of Chicago.

Assistant Professor H. F. DeBaggis of the University of Notre Dame has been appointed Head of the Department of Mathematics of Carroll College, Montana.

Dr. D. L. Fuller of the American Cyanamid Company has accepted a position as Director of Research with the Grace Chemical Research and Development Company, New York, New York.

Dr. George Gioumousis of the University of Wisconsin has accepted a position as a theoretical chemist with the Shell Development Company, Emeryville, California.

Dr. Stanley Katz of the Electro Data Corporation has accepted a position with the American Cyanamid Company, New York, New York.

Mr. M. L. Keedy, instructor of physics at the University of Nebraska, will serve as a science counselor for the coming year on a pilot study being conducted in Nebraska to determine ways and means of raising the level of mathematics and science teaching in the secondary schools. This study is sponsored by the American Association for the Advancement of Science and has received a grant from the Carnegie Foundation for this purpose.

Assistant Professor P. J. Kelly of the University of California, Santa Barbara College, has been promoted to an associate professorship.

Associate Professor E. S. Kennedy of the American University, Beirut, Lebanon, has been promoted to a professorship.

Mr. P. E. Klebe, Jr., has been appointed to an instructorship at the Polytechnic Institute of Brooklyn.

Professor A. S. Merrill, who has been both Dean of the Faculty and Vice-President of Montana State University, will continue as Vice-President.

Dr. W. A. Michael, Jr., of Northwestern University has accepted a position as applied science representative with International Business Machines Corporation, San Francisco, California.

Dr. John Raleigh of the University of Pennsylvania has been appointed to an assistant professorship at Lafayette College.

Mr. P. C. Rapp, formerly a dynamicist at Bell Aircraft Corporation, Niagara Falls, New York, is now an engineer at Cornell Aeronautical Laboratory, Buffalo, New York.

Mr. R. B. Rice, previously a senior physicist for the Phillips Petroleum Company, Bartlesville, Oklahoma, has a position as a senior research physicist for the Ohio Oil Company, Denver, Colorado.

Dr. G. F. Rose, recently a mathematician with the Sandia Corporation, Albuquerque, New Mexico, is a staff member of the Ramo-Wooldridge Corporation, Los Angeles, California.

Mr. J. A. Standerfer, formerly a teaching assistant at the University of Wisconsin, has a position as an aerophysics engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Mr. C. E. Stewart, previously a stress analyst for the Boeing Airplane Company, Seattle, Washington, is now a research engineer for the Armour Research Foundation, Illinois Institute of Technology.

Mr. A. V. Sylwester, formerly an instructor in the U. S. Air Force Weather Forecasting School, Chanute Air Force Base, Illinois, has been released from

active duty to accept a position as a mathematician at the Naval Ordnance Laboratory, Corona, California.

Mr. B. F. Whisler, previously a mathematician with the Army Map Service, Washington, D. C., has a position as a mathematical analyst with the Lockheed Aircraft Corporation, Burbank, California.

Professor C. R. White of the State Colored Normal, Industrial, Agricultural and Mechanical College of South Carolina has accepted a position as mathematician with the Weapon Systems Laboratory, Aberdeen Proving Ground, Maryland.

Assistant Professor R. L. Wilson of the University of Tennessee has a position as a senior aerophysics engineer with the Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Dean Fredrick Wood of the University of Nevada has retired with the title of Dean Emeritus.

Mr. B. H. Youell, Jr., previously a graduate assistant at West Virginia University, has been appointed to an assistant professorship at Waynesburg College.

Mr. J. S. Ayars, Jr., graduate student at Stanford University, died on February 3, 1956.

Assistant Professor Emeritus Harriet E. Glazier of the University of California at Los Angeles died on November 7, 1955. She was a charter member of the Association.

Professor Emeritus H. B. Leonard of the University of Arizona died on December 19, 1955. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE NEW EDITOR-IN-CHIEF

The Board of Governors of the Association has elected Professor Ralph D. James of the University of British Columbia as Editor-in-Chief of the AMERICAN MATHEMATICAL MONTHLY for a five-year period beginning January 1, 1957.

Professor James has served as a member of the Board of Governors from 1952 to 1955 as a Governor from the Pacific Northwest Section. He has also served as Vice-Chairman and Chairman of the Pacific Northwest Section. He is

an editor of the *Canadian Journal of Mathematics* and of the *Pacific Journal of Mathematics* and has served as an editor of the *American Journal of Mathematics*. He is a member of the Council of the American Mathematical Society and is a Vice-President of the Canadian Mathematical Congress.

After July 1, 1956, articles intended for publication in the MONTHLY should be sent to Professor James at this address: University of British Columbia, Vancouver 8, B.C., Canada.

H. M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 113 persons have been elected to membership by the Board of Governors on applications duly certified.

F. B. ALLEN, M.S. (Iowa) Math. Teacher, Lyons Township High School, LaGrange, Ill.

A. T. ANASTASIO, Student, Fordham University.

R. C. ANDERSON, SR., M.S. (Mississippi) Asst. Professor, McNeese State College.

B. H. ARNOLD, Ph.D. (Princeton) Asso. Professor, Oregon State College.

RIC ARUFFO, B.A. (Temple) Math., Philco Corp., Philadelphia, Pa.

W. F. BAIRD, JR., B.A. (Texas Christian) Tech. Analyst, Consolidated-Vultee Aircraft Corp., Fort Worth, Texas.

A. F. ANN BAKER, B.S. (Kentucky S.C.) Grad. Fellow, Fisk University.

WILFRED E. BARNES, Ph.D. (British Columbia) Instr., State College of Washington.

MRS. LIDA K. BARRETT, Ph.D. (Pennsylvania) Instr., University of Connecticut at Waterbury.

MRS. VIRGINIA L. BARTELL, M.Ed. (Oregon) Math. Teacher, Grant High School, Portland, Ore.

E. L. BATTISTE, B.S. (Illinois State Normal) Applied Science Representative, International Business Machines Corp., Oklahoma City, Okla.

T. E. BAYNHAM, JR., B.S. (Austin Peay) Grad. Asst., University of Tennessee.

JACQUES BAZINET, B.S. (Montreal) Grad. Student, University of Montreal.

LT. (J.G.) F. T. BIRTEL, M.S. (Notre Dame) Math. Instr., U. S. Navy Nuclear Power School, New London, Conn.

P. K. BLACKWELL, B.A. (Chicago) Milwaukee School of Engineering.

S. S. BLAKNEY, M.S. (Illinois) Instr., Grambling College.

A. H. BOWKER, Ph.D. (Columbia) Professor, Stanford University.

NEAL BRADLEY, Student, Georgia Institute of Technology.

R. L. BROCK, Ph.D. (Oregon S.C.) Res. Engr., Boeing Airplane Co., Seattle, Wash.

M. F. BRYN, B.S. (North Dakota Agric. C.) Instr., North Dakota Agricultural College.

J. F. BURRIS, B.S. (Southeastern S.C.) Grad. Asst., Oklahoma Agricultural and Mechanical College.

J. E. CALVERT, Student, San Jose State College.

D. F. CLAPP, S.B. (M.I.T.) Grad. Student, University of Technology.

E. N. CLARK, M.S. in E.E. (Minnesota) Electrical Engr., Pioneer Service and Engineering Co., Chicago, Ill.

W. R. COOPER, M.A. (Columbia) Teacher, Northeastern Academy, New York, N. Y.

L. J. CORIO, Student, City College of the City of New York.

M. D. COX, Student, Tennessee Polytechnic Institute.

T. H. M. CRAMPTON, Ph.D. (Indiana) Instr., Mount Holyoke College.

WARD CROWLEY, S.M. (Brown) Asst. Professor, State College of Washington.

W. S. CURRIE, M.A. (Mississippi) Math., Texas Company Research Lab., Bellaire, Texas.

H. J. CURTIS, Ph.D. (Illinois I.T.) Asst. Professor, University of Illinois, Chicago.

LT. COL. H. T. DARRACOTT, M.S. (Arkansas) Asst. Director of Research, Signal Corps Engineering Labs., Fort Monmouth, N. J.

M. D. DAVIS, M.A. (California) Grad. Student, University of California.

LUTHER DAWES, B.S. (Oklahoma City U.) Math. and Science Teacher, Jarman Junior High School, Midwest City, Okla.

W. S. DORN, Ph.D. (Carnegie) Supervisor, General Electric Co., Cincinnati, Ohio.

D. E. EDMONDSON, Ph.D. (Calif. I.T.) Asst. Professor, Southern Methodist University.

J. W. ELLIS, Ph.D. (Tulane) Asst. Professor, Florida State University.

H. J. ELLSWORTH, Partner, Fankhanel-Ellsworth Properties, Minneapolis, Minn.

ANGELINE W. EVANS, Student, Agnes Scott College.

H. C. FILGO, JR., Ph.D. (Rice) Asst. Professor, University of Alabama.

G. D. FINDLAY, M.A. (Cambridge) Lecturer, McGill University.

C. P. GADSDEN, M.S. (Tulane) Instr., Tulane University.

Z. T. GALLION, Ph.D. (Louisiana S.U.) Professor, Southwestern Louisiana Institute.

B. N. GOULET, M.S. (Catholic) Math., Ballistic Research Labs., Aberdeen, Md.

J. R. GRAY, M.A. (Loyola, Ill.) Teacher, Wright Junior College, Chicago, Ill.

W. R. HARLOW, M.A. (Cincinnati) Asst. Professor, University of Cincinnati.

R. J. HARVEY, A.B. (California) Grad. Student, University of California.

T. P. HIGGINS, Ph.D. (Texas) Res. Engr., Boeing Airplane Co., Seattle, Wash.

J. B. F. HOCEPIED, Ingenieur-Architecte (Ghent) Consulting Engr., Bruges, Belgium.

KLAUS HÖECHSMANN, Student, University of British Columbia.

BERNARD HOWARD, B.S. (William and Mary) Grad. Asst., Brown University.

W. J. KAMMERER, M.S. (Wisconsin) Grad. Student, University of Wisconsin.

NAOKI KIMURA, M.S. (Osaka U.) Grad. Fellow, Tulane University.

R. D. KING, Student, Georgia Institute of Technology.

TADASHI KISHI, M.A. (California) Math., University of California Radiation Lab., Livermore, Calif.

W. R. KNIGHT, B.A. (British Columbia) Grad. Asst., University of British Columbia.

CLARENCE LANE, M.S. (Purdue) Instr., Purdue University.

L. L. LASSEN, M.A. (Arizona S.C.) Asso. Professor, Arlington State College.

BERNADINE C. LAW, B.S. (Mount St. Scholastica) Grad. Fellow, St. Louis University.

LOUIS LEMIEUX, Professor, College Jean de Brebeuf, Montreal.

COL. HENRY LINSERT, M.A. (Pennsylvania S.U.) Instr., East Tennessee State College.

L. I. LOKOMOWITZ, B.E.E. (C.C.N.Y.) Jr. Engr., Philco Corp., Philadelphia, Pa.

E. F. LOW, JR., Ph.D. (Florida) Asst. Professor, University of Miami.

MRS. GRETA S. MACK, A.B. (Duke) Asst. Instr., University of Kansas.

BERNARD MALINA, M.S. (Northwestern) Instr., Wright Junior College, Chicago, Ill.

L. L. McNAIR, M.A. (Georgia) Instr., University of Georgia.

STELLA R. MIZELL, A.B. (North Carolina) Professor, Chowan College.

P. R. MONSON, B.S. (Utah State Agric.) Instr., Utah State Agricultural College.

AUGUSTA H. MORSE, M.A. (Radcliffe) Math. Teacher, Anna Head School, Berkeley, Calif.

H. A. MORTENSEN, Student, Utah State Agricultural College.

ABRAHAM NEMETH, M.A. (Columbia) Instr., University of Detroit.

C. C. NIELSEN, B.S. (Utah State Agric.) Instr., Utah State Agricultural College.

F. G. O'BRIEN, M.S. (Fordham) Asst. Professor, Fordham University.

R. O. PAN, Student, Case Institute of Technology.

EMANUEL PAPADOPULOS, B.A. (Sacramento S.C.) Jr. Engr., Boeing Airplane Co., Seattle, Wash.

D. W. PATTERSON, A.B. (Sacramento S.C.) Physicist, University of California Radiation Lab., Livermore, Calif.

ELAINE B. PAVELKA, B.A. (Northwestern) Grad. Student, Northwestern University.

R. Q. PETERSEN, Student, San Jose State College.

F. R. PLACHY, M.A. (Wyoming) Teacher, Center High School, Colo.

H. E. POKORNY, B.S. (Marquette) Instr., Milwaukee School of Engineering.

STANLEY PREISER, M.S. (N.Y.U.) Math., Nuclear Development Corp. of America, White Plains, N. Y.

R. F. QUANSTROM, JR., Student, Eastern Nazarene College.

A. T. RICE, A.M. (Boston U.) Math., United States Air Force, Washington, D. C.

D. W. ROBINSON, M.A. (Utah) Instr., Case Institute of Technology.

J. D. ROBINSON, Student, Siena College.

H. L. ROLF, M.A. (Vanderbilt) Instr., Vanderbilt University.

R. E. K. ROURKE, A.M. (Harvard) Head, Department of Mathematics, Kent School, Conn.

L. A. RUBEL, Ph.D. (Wisconsin) Instr., Cornell University.

H. J. SCHNELKER, Student, Purdue University, Fort Wayne, Ind.

P. A. SCHWEITZER, Student, Holy Cross College.

T. R. SHERROW, B.S. (Tulsa) Senior Clerk, Sinclair Oil & Gas Co., Tulsa, Okla.

P. C. SHIELDS, Student, Colorado College.

W. A. SILLARS, B.A. (Pennsylvania S.U.) Asst., Pennsylvania State University.

RONALD SIRCY, Student, Tennessee Polytechnic Institute.

SISTER MARY ALBERTA, A.M. (Loyola, Ill.) Head, Department of Mathematics, Central Catholic High School, Toledo, Ohio.

SISTER MARY CAROLINE WALLACE, M.S. (Marquette) Instr., St. Xavier College, Chicago, Ill.

SISTER MARY MATTHEW DONOVAN, Ed.D. (Houston) Asst. Professor, Sacred Heart Dominican College, Houston, Texas.

G. A. SOPP, M.S. in M.E. (Southern California) Res. Engr., Rocketdyne, Woodland Hills, Calif.

D. A. STEINBERG, B.A. (Kansas) Math., University of California Radiation Lab., Livermore, Calif.

E. J. STEWART, Ph.D. (Rice) Asso. Professor, U. S. Naval Postgraduate School.

RALPH SURASKY, B.S. (South Carolina) Grad. Asst., University of South Carolina.

A. M. SUTTON, B.A. (Harvard) Actuarial Student, Equitable Life Assurance Society, New York, N. Y.

PATRICIA A. TULLEY, A.B. (Vassar) Teacher, Kent School, Englewood, Colo.

P. J. USAVAGE, B.S. (Alliance) Instr., Chester High School, Pa.

R. J. WAGNER, B.S. (Lebanon Valley) Teaching Asst., Rutgers University.

J. H. WALTER, Ph.D. (Michigan) Instr., University of Washington.

R. L. WEBSTER, B.A. (Harvard) Field Representative, Addison-Wesley Publishing Co., Cambridge, Mass.

CHARLES WEISS, Student, Siena College.

ROSEMARY F. WILEY, A.M. (Chicago) Instr., University of Illinois, Chicago.

WALTER L. WILLIAMS, Student, University of Chicago.

G. L. WILSON, Student, University of California.

J. B. WILSON, M.S. (Cornell) Grad. Asst., University of Florida.

LT. H. A. YOUNG, A.B. (Harvard) Asst. Consultant Meteorologist, Air Research and Development Command, Baltimore, Md.

THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The eighteenth annual meeting of the Northern California Section of the Mathematical Association of America was held at Stanford University, Stanford, on January 14, 1956. Professor C. C. Torrance, Chairman of the Section, presided at both the morning and the afternoon sessions.

There were 105 persons in attendance at the meeting including 65 members of the Association.

At the business meeting the following officers were elected for the coming year: Chairman, Professor H. L. Alder, University of California, Davis; Vice-Chairman, Professor Harley Flanders, University of California, Berkeley; Secretary-Treasurer, Professor Roy Dubisch, Fresno State College, Fresno.

By invitation of the section, Professor Ivan Niven, University of Oregon and University of California, delivered an address at the morning session entitled *Irrational Numbers*. Abstract of this address follows:

This expository paper treated two parts of the subject: (1) a review of the classification, rational and irrational, algebraic and transcendental, of the commonly occurring numbers of mathematics, culminating in the generalized Lindemann theorem and the Gelfond-Schneider theorem; (2) the approximation of irrational numbers by rationals, concluding with the recent result of K. F. Roth on the best possible approximation of algebraic numbers by rationals. A new proof of the irrationality of the trigonometric functions for non-zero rational values of the arguments was given. The talk consisted of selections from the speaker's forthcoming Carus Monograph on the subject.

The following papers were presented:

1. *A curious trigonometric identity*, by Professor Raphael M. Robinson, University of California, Berkeley.

The formula $|\sin(x+iy)| = |\sin x + \sin iy|$ is valid for real x and y. This functional equation essentially characterizes the sine function.

2. *Calculations with random digits*, by Professor C. L. Perry, United States Naval Postgraduate School, Monterey.

Procedures for obtaining estimates to integrals by calculations using sequences of random digits were reviewed. This process, called the Monte Carlo method, is described in Applied Mathematics Series No. 12 (National Bureau of Standards), and in textbooks such as Householder's *Principles of Numerical Methods*, McGraw-Hill, 1953. Derivation of the expected value and variance for the Monte Carlo estimate for the value of a double integral was used for illustration of the method.

3. *Closed level curves*, by Professor H. A. Arnold, University of California, Davis.

In teaching complex variable, the level curves, $f(z)=k$, are usually assumed to be closed, although counter-examples such as $|f(z)| = |e^z| = e^x = k$ (a straight line here) are common. It is pointed out that if $f(z)$ is continuous on the unit disc $|z| \leq 1$ and holomorphic on its interior, $f(z)=0$ if and only if $z=0$; and if $f'(0) \neq 0$, then $|f(z)| = k$ is a simple closed analytic curve for z in $|z| \leq 1$ and $0 < k < \min f(z)$ on $|z| = 1$. It is pointed out that by considering $f(z)$ merely a local homeomorphism, similar conclusions may be drawn.

4. *A remark on the axiom of continuity*, by Professor C. M. Fulton, University of California, Davis.

Hilbert's plane axioms of incidence and the axioms of order are assumed as well as the postulate of Dedekind. It is shown that there exists at least one line through a given point which does not intersect a given line. If the axiom of continuity is omitted, all lines may intersect.

5. *Psychological factors in attitudes towards mathematics*, by Professors D. A. Norton and T. M. Poffenberger, University of California, Davis.

During the summer of 1955 at the University of California at Davis a preliminary survey was made of the factors contributing to an individual's attitude towards mathematics. The principal factors tentatively identified are the following: parents determine the initial attitudes of their children towards mathematics; the parents' expectation of the child's performance influences his achievement; the mathematics teachers have a strong effect on his further development.

6. *Formation of probability concepts*, by Professor C. C. Torrance, United States Naval Postgraduate School, Monterey.

This paper presented arguments for the following issues. (1) The basic addition and multiplication laws should be given *ab initio* for the general cases of dependent events and overlapping classifications. (Tests show that, even among trained mathematicians and scientists, the special forms of these laws are habitually misapplied to general situations because of the usual overemphasis of these special forms.) (2) The usual simple proofs of the basic laws can easily be made rigorous and general by introducing the concept of a *representative* sample of trials. (3) Standard functional notation should be used for conditional probabilities instead of the classical notation $Pr(E|F)$. This minimizes the difficulty of the transition from summation to integration.

7. *Some heuristic opportunities in the teaching of analytic geometry*, by Mr. C. M. Larsen, San Jose State College.

The point of view was advanced that heuristic teaching (as illustrated by G. Pólya in his book *How To Solve It*, Princeton University Press, Princeton, N. J., 1945) is desirable, but not easy to work into typically crowded courses. A few instances were then discussed where some major heuristic points can be conveniently worked into analytic geometry and calculus courses. These points included symmetry, generalization and specialization, and, especially, the sequence of leading questions: Do you know a related problem? Can you use its method? Can you use its result?

8. *Regular polygons with an odd number of sides*, by Professor V. E. Hoggatt, San Jose State College.

The question, as to whether or not three diagonals of a regular odd-sided polygon can be concurrent except at the vertices, was answered in the negative for $2n+1$ where its prime factors are greater than 11. This is a partial solution to problem 4444 of the advanced problem section of the MONTHLY (page 422, vol. 58, no. 6, June–July, 1952).

9. *The solution of simultaneous algebraic equations on an automatic digital computer*, by Professor B. J. Lockhart, United States Naval Postgraduate School, Monterey.

The solution of a system of four fourth order polynomial equations by various iterative methods on a CRC 102A digital computer is described. The practical requirement of obtaining convergence in an economically feasible length of time is the basic problem considered. The techniques described are well known as the gradient, composite gradient, successive searching and inversion of the Jacobian matrix methods.

10. *Unknowns, variables, and indeterminates*, by Professor Leonard Tornheim, University of California, Berkeley.

In the theory of polynomials and rational functions the symbol "x" plays three roles—unknown, variable, and indeterminate (or transcendental). Confusion occurs when these roles are not distinguished in topics such as cancellation and the remainder theorem.

C. D. OLDS, *Secretary*

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirty-third annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at McNeese State College, Lake Charles, Louisiana, on February 17 and 18, 1956. The Friday afternoon meeting was held in two concurrent sessions. Dean W. H. Bradford and Professor B. O. VanHook presided. Professor T. L. Reynolds presided at the evening session and at the Saturday morning session.

There were 124 persons registered including 51 members of the Association.

The following officers were elected for the coming year: Chairman, Dean W. H. Bradford, McNeese State College; Vice-Chairman for Mississippi, Professor R. D. Sheffield, Jr., University of Mississippi; Vice-Chairman for Louisiana, Professor Elsie T. Church, Northwestern State College; Secretary-Treasurer, Professor T. L. Reynolds, Millsaps College.

The invited speaker for the meeting was Professor J. C. Eaves of the University of Kentucky. His address on Friday evening was entitled "One, Two, Three, . . . ", and at the Saturday morning session, "A Matrix and a Number."

The following papers were presented:

1. *Report on the Summer Institute for Teachers of Collegiate Mathematics*, by Professor H. F. Schroeder, Louisiana Polytechnic Institute.

The speaker discussed the programs of the Summer Institutes for Teachers of Collegiate Mathematics and Teachers of Secondary Mathematics which were held at Oklahoma Agricultural and Mechanical College on June 13–July 23, 1955. These programs were sponsored by the National Science Foundation.

2. *Freshman mathematics programs*, by Professor F. A. Rickey, Louisiana State University.

The speaker presented a summary of the problems facing planners of courses in freshman mathematics with illustrations from the program at Louisiana State University.

3. *Mathematics in general education*, by Professor W. H. Spragens, University of Mississippi.

The speaker, who spent the year 1954–55 at the College of the University of Chicago as a Carnegie Intern in General Education, described the mathematics course in the curriculum there.

4. *Cubic inversion*, by Professor Elsie T. Church, Northwestern Louisiana College.

In this paper the speaker developed inversion using a cubic as the base curve and discussed the following: the inverses of various curves and points; illustrations of the effects of cubic inversion on the multiple points of a n-ic; invariants under the transformation; comparisons between cubic inversion and quadric inversion.

5. *The names of the curve of Agnesi*, by Reverend T. F. Mulcrone, St. Charles College.

The speaker quoted the pertinent passages from G. Grandi's note in the *Opere di Galileo Galilei* and his *Quadratura Circuli*, and M. G. Agnesi's *Instituzioni Analitiche*, and showed how the locus associated with the expression 'sinui verso'—the projection of a (circular) arc upon the diameter of the circle,—came eventually to be so badly designated, from mathematical and

etymological considerations, by the term "Witch" of Agnesi. It was noted also that the term "Witch" was a singularly inept one, because of its sinister implications, to predicate anything intended to honor a most honorable woman—the noted linguist, philosopher, mathematics writer, and hospital nun, Maria Gaetana Agnesi (1718–1799). The speaker recommended that English and American authors adopt the practice of the French (courbe d'Agnesi) and the Germans (Agnesische Kurve), writing simply "the curve of Agnesi."

6. *A construction relevant to the catenary and tractrix*, by Professor B. Ernest Mitchell, University of Mississippi.

The catenary and the tractrix are transcendental curves, but due to the fact that some of their elements are related algebraically certain constructions relative to the curves can be made, namely to draw their tangents at a given point and to find their radii of curvature there.

7. *Circles and the normal line: spheres and the normal plane*, by Professor V. B. Temple, Louisiana College.

The rectangular equations of two circles (spheres), each of radius r tangent to each other, and each tangent, one externally and the other internally, to the circle (sphere) of radius a with center at the origin, are respectively,

(1) $[x-(a\pm r)\cos\theta]^2+[y-(a\pm r)\sin\theta]^2=r^2$, (rolling circles),

(2) $[x-(a\pm r)\cos\alpha]^2+[y-(a\pm r)\cos\beta]^2+[z-(a\pm r)\cos\gamma]^2=r^2$, (rolling spheres),

where θ and α, β, γ are the direction angles of the line of centers. Subtracting equations (1) and (2) independently, we have

$$x\cos\theta+y\sin\theta=a, \quad \text{and} \quad x\cos\alpha+y\cos\beta+z\cos\gamma=a,$$

the normal line and the normal plane respectively.

8. *Points symmetric with respect to a fixed line in a plane*, by Professor J. T. Humphrey, Grambling College.

The case considered discusses the equations of a particular transformation. The properties of the transformation are exhibited as well as their application to certain curves. The discussion has been limited to cases most commonly met and to those which prove interesting.

The general line, the lines $x=0$, $y=0$, $x=y$, and $x=-y$ are chosen as axes of reflexion. Here we note that a point reflected through $y=0(Tx)$ then $x=0(Ty)$ is identical to its reflexion through the origin (T_0), *i.e.*, $T_yT_x=T_0$, similarly $T_xT_y=T_0$, $T_0T_y=T_x$, $T_yT_0=T_x$, $T_0T_x=T_y$, and $T_xT_0=T_y$.

The order in which the various topics are discussed is as follows: (1) definition; (2) co-ordinates of point pairs associated with the general case; (3) the lines $y=-x$, $y=x$, $y=0$, and $x=0$; (4) theorem: straight lines subjected to the transformation remain straight lines; (5) theorem: conics subjected to T_r remain invariant with respect to type; (6) theorem: if a curve in the plane is subjected to any two of the transformations T_x, T_y, T_0 the result will have the effect of the third in each case.

9. *Solution of heat equation on the electronic analog computer*, by Professor Margaret M. LaSalle, Southwestern Louisiana Institute.

The heat equation for a homogeneous medium of two dimensions

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

is solved by use of the electronic analog computer using the difference method. The error introduced by such a method is computed in a problem with nine stations. Considering symmetry, the set-up for the analog computer involving six amplifiers is given and the physical significance of the graphs so obtained is discussed. This problem was solved during a course at University of Michigan.

10. *A Pascal triangle for the coefficients of a polynomial*, by Professor R. A. Miller, University of Mississippi.

If $S(r_i, j, k)$ denotes the sum of the products, taken j at a time, of the first k elements of the set of n elements, r_i, with $S(r_i, 0, k)=1$ and $S(r_i, j, k)=0$, for $k<j$, it is observed that a simple recursion formula holds: $S(r_i, j, k)=S(r_i, j, k-1)+r_kS(r_i, j-1, k-1)$ for $j>1$, $k>1$. If $S(r_i, j, k)$ is the element in the jth row and kth column of a matrix and the elements of the jth row are multiplied by $(-1)^j$, the resulting elements of the kth column will be the coefficients of a polynomial with zeros $r_1, r_2, \cdots, r_k$. A Pascal triangle for the binomial coefficients results when the zeros are all equal to -1.

11. *On the matrix equation* $PX=Q$, by Professor T. J. Pignani, Loyola University.

This paper discusses conditions that must be imposed on P and Q so that the solution, $X=S$, of $PX=Q$ be a non-singular matrix, where P and Q are matrices with m rows and n columns, whose elements are real numbers. The paper contains preliminary results on the matrix equation $AYB=C$. Further it shows when certain conditions, to obtain the above results, are relaxed, then S is of rank not greater than a number determined by the rank of the coefficient matrices. The paper points out the application of the above results to interface problems.

12. *A formal derivation of finite sums*, by Professor S. R. Knox, Millsaps College.

By using the definition of the definite integral,

$$\int_a^b f(x)dx = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(\xi_i)(X_i - X_{i-1}),$$

a formal method of obtaining formulas for certain finite sums is demonstrated.

13. *The rise of a liquid*, by Professor P. K. Smith, Louisiana Polytechnic Institute.

The paper discussed certain functions derived from the rotation of a liquid in a smooth container. A general type container was first considered. The surface of such a container was obtained by revolving the curve $x^2=f(y)$ about the y axis. In this function $x^2-f(0)=0$ was assumed to have two real, non-zero roots. Further, it was assumed that $dy/dx>0$ for all values of y. A fundamental relationship was derived which gave the angular velocity of the rotative fluid as a function of the initial fluid depth, σ, the rise, λ, caused by a given angular velocity, and the function $f(\lambda)$. Then several conclusions were drawn from this fundamental equation.

14. *Some probabilities*, by Professor T. A. Bickerstaff, University of Mississippi.

From an infinite universe $\{x\}$ with continuous density function $f(x)$, with no parameters known, a sample (x_i), $i=1, 2, \cdots, n$ was taken. To the order statistics, x_i, the probability integral transform $p_a=\int_{-\infty}^{x_a} f(x)dx$ was applied. With order preserved, the well-known Beta distribution of the order statistic, p_a, was pointed out and the super Beta function of the joint distribution of two order statistics was derived. These yielded a variety of probabilities and confidence limits for order parameters. Methods of evaluating the incomplete integrals were indicated.

Z. L. LOFLIN, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 15–16, 1957.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, Concordia College, Moorhead, October 6, 1956.
MISSOURI
NEBRASKA
NORTHEASTERN, University of Connecticut, Storrs, November 24, 1956.
NORTHERN CALIFORNIA
OHIO
OKLAHOMA, Oklahoma City University, October 26, 1956.
PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.
PHILADELPHIA, November 24, 1956.
ROCKY MOUNTAIN
SOUTHEASTERN, Emory University, Emory University, Georgia, March 15–16, 1957.
SOUTHERN CALIFORNIA, San Diego State College, Spring, 1957.
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

THE CARUS MATHEMATICAL MONOGRAPHS

These Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and scientific workers in other fields. One copy of each Monograph may be purchased by members of the Association for $1.75 each. Additional copies and copies for non-members of Monographs 1–8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois. In the case of Monographs 9 and 10, additional copies and copies for non-members must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, N. Y. These numbers have been issued to date:

No. 1. *Calculus of Variations* by G. A. Bliss, xii+189 pages.
No. 2. *Analytic Functions of a Complex Variable* by D. R. Curtiss, ix+173 pages.
No. 3. *Mathematical Statistics* by H. L. Rietz, ix+181 pages.
No. 4. *Projective Geometry* by J. W. Young, ix +185 pages.
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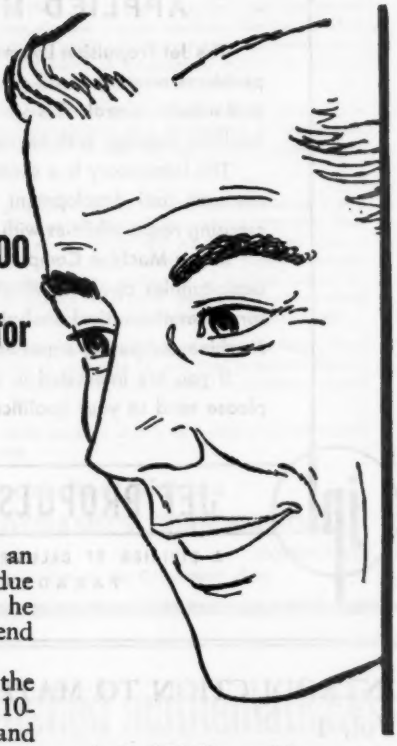

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